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Compactness, metrizability, and Baire isomorphism

By E. R. LORCH and HING TONG in New York (N. Y., U.S.A.)

To Béla Sz.-Nagy on his sixtieth birthday

Any comparative study of compact spaces naturally begins with a suitable characterization of compact metric spaces. A topological characterization of this type is given in the present paper. The characterization is based on a result concerning Baire isomorphisms between two spaces. Such isomorphisms between compact metric spaces have been studied in some detail by the authors ([3], [4], [5], [6]).

All spaces considered are separated. Given such a space $\mathbf{X}$, a set $\mathbf{M} \subset \mathbf{X}$ is the zeroset of a continuous real-valued function f in case $\mathbf{M} = \{x : x \in \mathbf{X}, f(x) = 0\}$. The family of Baire sets on $\mathbf{X}$ is the smallest family which contains all zerosets of continuous functions and which is closed under complementation, denumerable union, and denumerable intersection. The family of Baire functions is the smallest family of bounded real-valued functions which contains all bounded continuous real-valued functions and which is closed under the operation of the taking of pointwise limits of sequences of functions. A Baire isomorphism $\Phi : \mathbf{X} \to \mathbf{Y}$ is a bijection such that for any Baire set $\mathbf{M} \subset \mathbf{X}$, $\Phi(\mathbf{M})$ is a Baire set in $\mathbf{Y}$; and such that for any Baire set $\mathbf{N} \subset \mathbf{Y}$, $\Phi^{-1}(\mathbf{N})$ is a Baire set in $\mathbf{X}$. A Baire function is Baire measurable and any Baire measurable function is a Baire function.

A family of real-valued functions defined on $\mathbf{X}$ is said to separate points in case for each $x_1, x_2 \in \mathbf{X}$, $x_1 \neq x_2$, there exists a function f of the family such that $f(x_1) \neq f(x_2)$. Although the role of separation of points for families of continuous functions is universally known, results for separation by noncontinuous functions are less common. Our first theorem concerns such a situation.

Theorem A. *Let $\mathbf{X}$ be a compact space. Then $\mathbf{X}$ is metrizable if and only if there exists a denumerable set of Baire functions $\{\varphi_n\}$ which separates the points of $\mathbf{X}$.*

Proof. If $\mathbf{X}$ is compact and metrizable then there exists a denumerable set of continuous functions $\{\varphi_n\}$, hence, a fortiori, Baire functions, which separate points.

Now let $\{\varphi_n\}$ be a denumerable family of Baire functions which separates points.

Consider one of the functions of the family. Call it φ. We shall say that a set $\mathscr{F}$ of continuous functions is an *ancestral family* for φ in case the smallest family of Baire functions containing $\mathscr{F}$ and closed under the process of the taking of pointwise limits of sequences of functions contains φ. Thus, one ancestral family for φ is the family of all continuous real-valued functions on $\mathbf{X}$. In general, there are many other ancestral families for φ.

It may easily be seen that if φ is a Baire function, there exists at least one denumerable ancestral family for φ. This can be proved by transfinite induction on the order α of the Baire function. If $\alpha=0$, then φ is continuous and $\{\varphi\}$ is an ancestral family for φ. Suppose now that φ is of Baire class α and that the result is valid for every Baire function of order $\beta<\alpha$. Suppose $\varphi(x) = \lim\limits_{n\to\infty} \varphi_n(x)$, where φ_n is a Baire function of class $\beta_n<\alpha$. If $\mathscr{G}_n$ is a denumerable ancestral family for φ_n, then $\mathscr{G} = \bigcup\limits_{n=1}^{\infty} \mathscr{G}_n$ is a denumerable ancestral family for φ.

Going back to the proof of the theorem, let $\mathscr{F}_n$ be a denumerable ancestral family of continuous functions for φ_n. Let $\mathscr{F} = \bigcup\limits_{n=1}^{\infty} \mathscr{F}_n$. Then $\mathscr{F}$ is a denumerable family of continuous functions. We write $\mathscr{F}=\{f_1, f_2, \ldots\}$.

It is clear that the functions of $\mathscr{F}$ separate points. Indeed, if for $x_1 \neq x_2$, $f(x_1)= =f(x_2)$ for each function in $\mathscr{F}$, then for any Baire function φ generated from $\mathscr{F}$, one has $\varphi(x_1)=\varphi(x_2)$. The hypothesis of Theorem A thus implies that $\mathscr{F}$ separates points. Now, let $\Xi: \mathbf{X} \to \mathbf{R}^{\omega}$ be the map of $\mathbf{X}$ into the denumerable product of $\mathbf{R}$, given by $\Xi(x)=(f_1(x), f_2(x), \ldots)$. Then Ξ is a continuous bijection of $\mathbf{X}$ onto the subset $\mathbf{Y}=\Xi(\mathbf{X})$ of $\mathbf{R}^{\omega}$. Since $\mathbf{X}$ is compact, so is $\mathbf{Y}$. Since Ξ is a continuous bijection from one compact space to another, it is bicontinuous. Thus Ξ is a homeomorphism between $\mathbf{X}$ and the metric space $\mathbf{Y}$. Hence $\mathbf{X}$ is metrizable.

There is a companion to Theorem A which concerns Baire sets instead of Baire functions. We state it below. Note that a collection $\{\mathbf{M}_\alpha\}$ of subset of a set X is said to separate the points of $\mathbf{X}$ providing that for every pair $x, y \in X$, $x \neq y$, there exists a set $\mathbf{M}_\alpha$ such that precisely one of the points, x, y belongs to $\mathbf{M}_\alpha$.

Corollary B. *Let* $\mathbf{X}$ *be a compact space. Then* $\mathbf{X}$ *is metrizable if and only if there exists a denumerable family* $\{\mathbf{M}_n\}$ *of Baire sets which separates the points of* $\mathbf{X}$.

Let χ_n be the characteristic function of $\mathbf{M}_n$. Then χ_n is a Baire function and the denumerable family $\{\chi_n\}$ separates the points of $\mathbf{X}$. Thus the preceding theorem applies. It should be noted that in case the family of Baire sets generates all the Baire sets, then, obviously, it separates points.

An important consequence of Theorem A will now be stated.

Corollary C. *If* $\mathbf{X}$ *is compact and is Baire isomorphic to a compact metric space* $\mathbf{Y}$, *then* $\mathbf{X}$ *is metrizable.*

Proof: Let $\Psi:\mathbf{X}\to\mathbf{Y}$ be the given Baire isomorphism. Let $\{g_1, g_2, \dots\}$ be a denumerable family of real-valued continuous functions defined on $\mathbf{Y}$ which separate the points of $\mathbf{Y}$. Define $\varphi_n = g_n \circ \Psi$, $n=1, 2, \dots$. Then φ_n is Baire measurable and hence a Baire function defined on $\mathbf{X}$ and the denumerable family $\{\varphi_1, \varphi_2, \dots\}$ separates the points of $\mathbf{X}$. Thus by Theorem A, $\mathbf{X}$ is metrizable.

A result including Corollary C is given by Jayne [1] for spaces $\mathbf{X}$ which enjoy certain properties in their beta compactification. The proof is non-elementary.

We introduce now a property that a topological space $\mathbf{X}$ may possess which plays a dominant role in what follows.

Property D. *There exits for each $x\in\mathbf{X}$ a sequence $(\mathbf{V}_n(x))$ of open neighborhoods of x such that if (x_n) is any sequence of points in $\mathbf{X}$, then $\bigcap_{n=1}^{\infty} \mathbf{V}_n(x_n)$ contains at most one point.*

Theorem E. *Let $\mathbf{X}$ be compact. Then $\mathbf{X}$ is metrizable if and only if $\mathbf{X}$ possesses property* D.

Proof. Suppose $\mathbf{X}$ is metric. For each x, let $\mathbf{V}_n(x)$ be the open sphere of radius $\frac{1}{n}$, center x. Then for any sequence (x_n), the intersection $\bigcap_{n=1}^{\infty} \mathbf{V}_n(x_n)$ contains at most one point. Hence $\mathbf{X}$ has property D.

We now assume that $\mathbf{X}$ has property D. We shall show that $\mathbf{X}$ is a Baire isomorphic to a compact metric space $\mathbf{Y}$ and is therefore metrizable by Corollary C.

For each $x\in\mathbf{X}$, let φ_x be a real-valued continuous function, $\varphi_x:\mathbf{X}\to[0, 1]$, such that $\varphi_x(x)=0$ and $\varphi_x(y)=1$ for $y\in\mathbf{X}-\mathbf{V}_1(x)$. Let

$$\mathbf{V}_1^0(x)=\left\{u\colon u\in\mathbf{X} \quad\text{and}\quad \varphi_x(u)<\frac{1}{2}\right\};\quad \mathbf{F}_1(x)=\left\{u\colon u\in\mathbf{X}\quad\text{and}\quad \varphi_x(u)\leqq\frac{1}{2}\right\}.$$

Thus $\mathbf{V}_1^0(x)\subset\mathbf{F}_1(x)\subset\mathbf{V}_1(x)$. The family $\{\mathbf{V}_1^0(x):x\in\mathbf{X}\}$ is an open cover for $\mathbf{X}$. Since $\mathbf{X}$ is compact, there exists a finite subcover in this family consisting, say, of $\{\mathbf{V}_1^0(x_1), \mathbf{V}_1^0(x_2), \dots, \mathbf{V}_1^0(x_{n_1})\}$. Thus $\{\mathbf{F}_1(x_1), \mathbf{F}_1(x_2), \dots, \mathbf{F}_1(x_{n_1})\}$ is also a cover of $\mathbf{X}$. We change the notation slightly and write $\{\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_{n_1}\}$ for this cover. Thus

$$\mathbf{X} = \mathbf{F}_1\cup\mathbf{F}_2\cup\dots\cup\mathbf{F}_{n_1}.$$

Set $\mathbf{F}_n=\emptyset$ for $n>n_1$. Then $\mathbf{X}=\bigcup_{n=1}^{\infty}\mathbf{F}_n$. Now let $\mathbf{F}_1^*=\mathbf{F}_1$ and $\mathbf{F}_k^*=\mathbf{F}_k-(\mathbf{F}_1\cup\dots\cup\mathbf{F}_{k-1})$ for $k>1$. Thus $\mathbf{X}=\bigcup_{k=1}^{\infty}\mathbf{F}_k^*$ and the sets in the representation are disjoint.

Since the sets $\mathbf{F}_n$ are zerosets, $\mathbf{F}_1 \cup \cdots \cup \mathbf{F}_k$ is also a zero set, $k=1, 2, \ldots$. Thus the set $\mathbf{F}_k^* = \mathbf{F}_k - (\mathbf{F}_1 \cup \cdots \cup \mathbf{F}_{k-1})$ is a denumerable union of zerosets. Anticipating an inductive proceedure, we write k_1 instead of k. Thus for $k_1 > 1$ we have $\mathbf{F}_{k_1}^* = \bigcup_{r=1}^{\infty} \mathbf{H}_{k_1 r}$ where $\mathbf{H}_{k_1 r}$ is a zero set.

If $\mathbf{K}$ is any zeroset (hence compact), one may cover $\mathbf{K}$ by a finite collection of zerosets each of which lies inside the neighborhood $\mathbf{V}_2(x)$ for some point x. In particular, for the set $\mathbf{F}_1^*$, we have $\mathbf{F}_1^* = \bigcup_{k_2=1}^{\infty} \mathbf{F}_{1k_2}$ where the sets $\mathbf{F}_{1k_2}$ are zerosets of this type ($=\emptyset$ for large k_2).

We apply to the set $\mathbf{H}_{k_1 r}$ the argument just given for $\mathbf{K}$ and find a finite collection of zerosets, each contained in some $\mathbf{V}_2(x)$, whose union is $\mathbf{H}_{k_1 r}$. The totality of these sets for $r=1, 2, \ldots$ is then enumerated and labelled $\mathbf{F}_{k_1 k_2}$; $k_2=1, 2, \ldots$ and we obtain for $\mathbf{F}_{k_1}^*$ the expression $\mathbf{F}_{k_1}^* = \bigcup_{k_2=1}^{\infty} \mathbf{F}_{k_1 k_2}$ where each $\mathbf{F}_{k_1 k_2}$ is a zeroset lying inside some $\mathbf{V}_2(x)$.

Proceeding inductively we obtain for each integer $k_1 \geqq 1, k_2 \geqq 1, \ldots$ sets

$$\mathbf{F}_{k_1} \supset \mathbf{F}_{k_1}^* \supset \mathbf{F}_{k_1 k_2} \supset \mathbf{F}_{k_1 k_2}^* \supset \mathbf{F}_{k_1 k_2 k_3} \supset \mathbf{F}_{k_1 k_2 k_3}^* \supset \ldots$$

satisfying the relations:

$$\mathbf{X} = \bigcup_{k_1=1}^{\infty} \mathbf{F}_{k_1}; \quad \bigcup_{k_1=1}^{n} \mathbf{F}_{k_1} = \bigcup_{k_1=1}^{n} \mathbf{F}_{k_1}^*;$$

the sets $\mathbf{F}_{k_1}^*$ are disjoint;

$$\mathbf{F}_{k_1}^* = \bigcup_{k_2=1}^{\infty} \mathbf{F}_{k_1 k_2}; \quad \bigcup_{k_2=1}^{n} \mathbf{F}_{k_1 k_2} = \bigcup_{k_2=1}^{n} \mathbf{F}_{k_1 k_2}^*;$$

the sets $\mathbf{F}_{k_1 k_2}^*$ are disjoint, and so on. Furthermore, each set $\mathbf{F}_{k_1 k_2 \ldots k_n}$ lies inside the set $\mathbf{V}_n(x)$ for some x. Also, each set $\mathbf{F}_{k_1 \ldots k_n}$ is a zeroset and each set $\mathbf{F}_{k_1 \ldots k_n}^*$ is a denumerable union of zerosets. Clearly we have $\bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n} = \bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n}^*$.

Note that if x is given, then there exists a unique sequence of the type $(\mathbf{F}_{k_1}^*, \mathbf{F}_{k_1 k_2}^*, \ldots)$ such that x belongs to each set of the sequence. Note also that if $(k_1, k_2, \ldots)$ is any sequence of positive integers, then the intersection $\bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n}^*$ is non-empty if and only if each of the sets $\mathbf{F}_{k_1 \ldots k_n}^*$ is non-empty. This results from the fact that $\bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n}^* = \bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n}$ and from compactness. Finally, since $\mathbf{F}_{k_1 \ldots k_n}^* \subset \subset \mathbf{V}_n(x_n)$ for some point x_n, $\bigcap_{n=1}^{\infty} \mathbf{F}_{k_1 \ldots k_n}^* \subset \bigcap_{n=1}^{\infty} \mathbf{V}_n(x_n)$ and by Property D, this intersection contains at most one point. In recapitulation, the intersection

$\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1\dots k_n}$ consists of precisely one point if and only if each set $\mathbf{F}^*_{k_1\dots k_n}$ is non-empty.

As usual, denote the set of all sequences of positive integers $k=(k_1, k_2, \dots)$ by $\mathbf{N}^{\mathbf{N}}$. The topology and metric character of $\mathbf{N}^{\mathbf{N}}$ play a very important role in what follows. Each $x \in \mathbf{X}$ has associated to it a unique sequence $(k_1, k_2, \dots)$ such that $\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1\dots k_n}=\{x\}$. Thus this defines a map

$$\Phi: \mathbf{X} \to \mathbf{N}^{\mathbf{N}},$$

where $\Phi(x)=(k_1, k_2, \dots)$.

It will be shown that Φ is a bijection from $\mathbf{X}$ to $\Phi(\mathbf{X})$. Furthermore, it will be shown that $\Phi(\mathbf{X})$ is a closed subset of $\mathbf{N}^{\mathbf{N}}$ and that Φ is a Baire isomorphism.

If $\Phi(x)=\Phi(x')=(k_1, k_2, \dots)$, then x and x' both belong to the set $\mathbf{F}^*_{k_1\dots k_n}$, $n=1, 2, \dots$, hence both belong to $\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1\dots k_n}$. The latter consists of one point by Property D. Thus Φ is a bijection onto $\Phi(\mathbf{X})$.

Let $k \in \mathbf{N}^{\mathbf{N}}$, $k \notin \Phi(\mathbf{X})$. Suppose $k=(k_1, k_2, \dots)$. It is easy to see that $\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1\dots k_n}=\emptyset$. In fact, if the intersection were not empty, it would consist of a single point x by Property D and one would have $\Phi(x)=k$. Now, by compactness and monotonicity if $\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1\dots k_n} = \bigcap_{n=1}^{\infty} \mathbf{F}_{k_1\dots k_n} = \emptyset$, then there exists n_0 such that $\mathbf{F}^*_{k_1\dots k_n}=\emptyset$, $n \geqq n_0$. This implies that the neighborhood of k consisting of the points $(k_1, \dots, k_{n_0}, l_{n_0}+1, \dots)$ where l_{n_0+j} is arbitrary, $j=1, 2, \dots$, does not intersect $\Phi(\mathbf{X})$. In other words, $\Phi(\mathbf{X})$ is closed in $\mathbf{N}^{\mathbf{N}}$.

Next we show that Φ^{-1} is a continuous map. Let $k=(k_1, k_2, \dots)$ be a point of $\Phi(\mathbf{X})$ and suppose that $\Phi^{-1}(k)=x$. Let $\mathbf{V}(x)$ be any neighborhood of x. Then since $\bigcap_{n=1}^{\infty} \mathbf{F}^*_{k_1\dots k_n}=\{x\}$, there exists by the compactness of X an integer n_0 such that $\mathbf{F}^*_{k_1\dots k_{n_0}} \subset \mathbf{F}_{k_1\dots k_{n_0}} \subset \mathbf{V}(x)$. Let $\mathbf{W}(k)=\mathbf{W}(k, n_0)=\{l=(l_1, l_2, \dots): l \in \Phi(\mathbf{X}),\ l_1=k_1, \dots$ $\dots, l_{n_0}=k_{n_0};\ l_{n_0+r} \in \mathbf{N},\ r=1, 2, \dots\}$. Then $\mathbf{W}(k)$ is an open neighborhood of k. Also, $\Phi^{-1}(\mathbf{W}(k)) \subset \mathbf{V}(x)$, since $\Phi^{-1}(\mathbf{W}(k))=\mathbf{F}^*_{k_1\dots k_{n_0}}$. Thus Φ^{-1} is continuous. It follows that Φ maps open sets into open sets. Thus Φ maps cozerosets into open sets (of a metric space) and hence Φ maps Baire sets into Baire sets.

Finally, we show that Φ^{-1} transforms Baire sets into Baire sets. Note first that the family of all sets $\mathbf{W}(k, n_0)$ where k is arbitrary and $n_0=1, 2, \dots$, is a denumerable base for the topology of $\Phi(\mathbf{X})$. Now $\Phi^{-1}(\mathbf{W}(k, n_0))=\mathbf{F}^*_{k_1\dots k_{n_0}}$. This set is a denumerable union of zerosets in $\mathbf{X}$. Thus if $\mathbf{G}$ is any open set in the metric space $\Phi(\mathbf{X})$, $\Phi^{-1}(\mathbf{G})$ is a Baire set in $\mathbf{X}$.

The above paragraphs prove that Φ is a Baire isomorphism. Also, $\Phi(X)$ is a complete separable metric space. If the cardinality of $\Phi(X)$ is $\mathbf{c}$ (cardinality of the continuum), then $\Phi(X)$ is Baire isomorphic to [0, 1], a compact metric space [2, p. 358]. If the cardinality of $\Phi(X)$ is $\aleph_0$, it is obviously Baire isomorphic to some compact metric space. Thus, in either case, X is metrizable by Corollary C. (For simplicity of statement, we invoke above the very general classical theorem concerning the Baire isomorphism. It would have been simple in our case to avoid the theorem and to construct the needed isomorphism directly.)

A different proof of Theorem E could be obtained by using the Urysohn metrization theorem. However, the development presented above will be used on subsequent occasions.

Added by the authors on May 2, 1973: As indicated in our introduction, the purpose of the paper is to provide a procedure for "grading" compact spaces. In this grading, the metric spaces occupy the position of simplest structure because they have Property D. The refinement procedure necessary to handle the non-metric case will be developed on another occasion. We wish to point out that a characterization of metric compact spaces has been given by V. Sneider, Continuous images of Souslin and Borel sets. Metrization theorems, *Doklady Akad. Nauk S.S.S.R.* (N.S.) **50** (1945), 77—79; *M.R.* 14—782.

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COLUMBIA UNIVERSITY
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On products of Toeplitz operators

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Dedicated to Béla Sz.-Nagy on his sixtieth birthday

1. Introduction

Brown and Halmos show in [2] that if f and g are functions in L^∞ of the unit circle and T_f and T_g are the corresponding Toeplitz operators on H^2, then for the equality $T_f T_g = T_{fg}$ to hold it is necessary and sufficient that either $\bar{f}$ or g belong to H^∞. The sufficiency of the preceding condition has been recognized since Toeplitz operators were first studied; it forms the basis for the Wiener—Hopf factorization technique. The necessity of the condition tells us that the equality $T_f T_g = T_{fg}$ is rather special.

It is much less special, however, for the difference $T_f T_g - T_{fg}$ to be compact, and this circumstance has been useful in the spectral analysis of certain Toeplitz operators. Coburn [3] has shown that $T_f T_g - T_{fg}$ is compact if either f or g is continuous, and Gohberg and Krupnik [9] have shown that $T_f T_g - T_{fg}$ is compact if f and g are both piecewise continuous and have no common discontinuities. In the present paper, a sufficient condition for the compactness of $T_f T_g - T_{fg}$ is presented which contains both of the conditions just mentioned.

For f and g in L^∞ and λ a point on the unit circle, we define

$$\operatorname{dist}_\lambda (f, g) = \operatorname{ess.} \limsup_{\substack{z \to \lambda \\ |z| = 1}} |f(z) - g(z)|.$$

If we extend f and g harmonically into the unit disk by means of Poisson's formula, we can also write

$$\operatorname{dist}_\lambda (f, g) = \limsup_{\substack{z \to \lambda \\ |z| < 1}} |f(z) - g(z)|.$$

For f in L^∞ we define

$$\operatorname{dist}_\lambda (f, H^\infty) = \inf \{\operatorname{dist}_\lambda (f, h): \ h \in H^\infty\}.$$

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A simple normal families argument shows that there exists an h in H^∞ such that $\mathrm{dist}_\lambda(f, h) = \mathrm{dist}_\lambda(f, H^\infty)$.

Theorem. *If f and g are in L^∞, and if for each λ on the unit circle either* $\mathrm{dist}_\lambda(\bar{f}, H^\infty)=0$ *or* $\mathrm{dist}_\lambda(g, H^\infty)=0$, *then $T_f T_g - T_{fg}$ is compact.*

The proof, which rests ultimately on Coburn's condition, is given in Section 2. R. G. Douglas [4, Corollary 7. 52] has found an alternative proof which uses the theory of C^*-algebras.

The theorem says, roughly, that if the condition in the Brown—Halmos theorem holds locally, then the equality $T_f T_g = T_{fg}$ holds to within a compact perturbation. Because the condition in the Brown—Halmos theorem is a necessary as well as a sufficient one, it is natural to conjecture the converse of the above theorem. The converse, however, is false, as is shown by a counter example in Section 3. Section 4 contains a partial converse of the theorem.

2. Proof of the theorem

For f in L^∞, the function $\mathrm{dist}_\lambda(f, H^\infty)$ is upper semicontinuous with respect to λ. This function therefore attains a maximum on the unit circle.

Lemma. *If f is in L^∞, then* $\mathrm{dist}(f, H^\infty + C) = \max\{\mathrm{dist}_\lambda(f, H^\infty) : |\lambda| = 1\}$.

Here, C denotes the space of continuous complex valued functions on the unit circle. The lemma is an immediate consequence of a result of Bishop and Glicksberg [8, p. 419] on sets of antisymmetry for function algebras (see [6] for a fuller explanation). To keep this paper as elementary as possible, a simple direct proof of the lemma is provided in Section 5.

Let f and g satisfy the hypotheses of the theorem. Choose $\varepsilon > 0$, and let A and B be the sets of points λ on the unit circle where $\mathrm{dist}_\lambda(\bar{f}, H^\infty) \geqq \varepsilon$ and $\mathrm{dist}_\lambda(g, H^\infty) \geqq \varepsilon$, respectively. The sets A and B are then closed and disjoint. Choose a nonnegative function u in C such that $u \leqq 1$, $u=0$ on A, and $u=1$ on B. Let $v = 1-u$. By Coburn's condition we have (letting $K_1, K_2, \ldots$ denote compact operators),

$$(1)\qquad T_f T_g = T_f T_u T_g + T_f T_v T_g = (T_{fu} + K_1) T_g + T_f (T_{vg} + K_2) = T_{fu} T_g + T_f T_{vg} + K_3.$$

Since $\mathrm{dist}_\lambda(vg, H^\infty) < \varepsilon$ for all λ, it follows from the lemma that $\mathrm{dist}(vg, H^\infty + C) < \varepsilon$. Hence, we can write $vg = h + w + \varphi$ where h is in H^∞, w is in C, and $\|\varphi\|_\infty < \varepsilon$. Because $T_f T_h = T_{fh}$ and $T_f T_w = T_{fw} + K_4$ (by Coburn's condition), we have

$$(2)\qquad T_f T_{vg} - T_{fvg} = T_f T_\varphi - T_{f\varphi} + K_4 = S_1 + K_4,$$

where $\|S_1\| \leqq \|f\|_\infty \|\varphi\|_\infty + \|f\varphi\|_\infty \leqq 2\varepsilon \|f\|_\infty$. Exactly the same reasoning gives

$$(3)\qquad T_{fu} T_g - T_{fug} = S_2 + K_5,$$

where $\|S_2\| \leqq 2\varepsilon \|g\|_\infty$. Combining (1)—(3), we obtain

$$T_f T_g = T_{fug} + T_{fvg} + S_1 + S_2 + K_6 = T_{fg} + S_1 + S_2 + K_6.$$

Hence, the distance of $T_f T_g - T_{fg}$ from the set of compact operators is at most $\|S_1\| + \|S_2\| \leqq 2\varepsilon(\|f\|_\infty + \|g\|_\infty)$. As ε can be chosen arbitrarily small, it follows that $T_f T_g - T_{fg}$ is compact. The theorem is proved.

3. Counter example to the converse

To obtain the desired counter example, we take $g = \bar{\psi}$ where ψ is the inner function $\exp\left(\frac{z+1}{z-1}\right)$. We take for f a real function in L^∞ with the following properties: (i) f is continuous except at $z=1$; (ii) $f/(1-z)$ is in L^2; (iii) f is not in $H^\infty + C$. We defer the construction of f until later. We note that $\operatorname{dist}_1(f, H^\infty) > 0$ by the lemma in Section 2. Also, it is easy to see that $\operatorname{dist}_1(g, H^\infty) = 1$. The condition of the theorem therefore fails at $\lambda = 1$. We show that, nevertheless, the operator $T_f T_g - T_{fg}$ is compact.

Because

$$(T_f T_g - T_{fg}) T_\psi = T_f T_{\bar{\psi}} T_\psi - T_{f\bar{\psi}} T_\psi = T_f T_{\bar{\psi}\psi} - T_{f\bar{\psi}\psi} = T_f - T_f = 0,$$

the operator $T_f T_g - T_{fg}$ annihilates the subspace ψH^2. Hence, it will be enough to show that the restriction of $T_f T_g - T_{fg}$ to $H^2 \ominus \psi H^2$ is compact. Also, the operator $T_g = T_{\bar{\psi}}$ annihilates $H^2 \ominus \psi H^2$, so we need only show that the restriction of T_{fg} to $H^2 \ominus \psi H^2$ is compact. We shall show that, actually, the transformation from $H^2 \ominus \psi H^2$ into L^2 of multiplication by fg is compact. Because multiplication by g sends $H^2 \ominus \psi H^2$ isometrically onto $\bar{\psi} H^2 \ominus H^2$, this amounts to showing that the transformation of $\bar{\psi} H^2 \ominus H^2$ into L^2 of multiplication by f is compact. Let S denote the latter transformation.

To prove the compactness of S, we introduce the isometry V of L^2 of the circle onto $L^2(-\infty, \infty)$ defined by

$$(Vh)(x) = \pi^{-1/2}(x+i)^{-1} h\left(\frac{x-i}{x+i}\right).$$

This isometry transforms the operator on L^2 of multiplication by f into the operator on $L^2(-\infty, \infty)$ of multiplication by $\varphi(x) = f\left(\frac{x-i}{x+i}\right)$. From (ii) it follows that φ is in $L^2(-\infty, \infty)$. Let W be the Fourier—Plancherel transformation of $L^2(-\infty, \infty)$ onto itself. Then W transforms the operator on $L^2(-\infty, \infty)$ of multiplication by φ into the operator of convolution with the square-integrable function $k = (2\pi)^{1/2} W\varphi$. Now it is easy to check that the combined transformation $U = WV$ sends $\bar{\psi} H^2 \ominus H^2$

onto $L^2(-1,0)$ (regarded as the subspace of functions in $L^2(-\infty,\infty)$ vanishing off $(-1,0)$). Hence U takes S into the transformation of $L^2(-1,0)$ into $L^2(-\infty,\infty)$ of convolution with k, that is, into the integral operator with kernel $K(x,y)=$ $=k(x-y)\ (-\infty<x<\infty,\ -1<y<0)$. The square-integrability of k implies the square-integrability of K, so the integral operator in question is a Hilbert—Schmidt operator. Therefore S is compact, as desired.

It remains to construct a function f with the required properties. For this we employ the notion of mean oscillation.

Let m denote normalized Lebesgue measure on the unit circle. For f in L^1 of the circle and I a subarc of the circle, define $av_I f=m(I)^{-1}\int_I f\,dm$ and

$$M(f,I)=m(I)^{-1}\int_I |f-av_I f|\,dm.$$

Further, define

$$M_r(f)=\sup\{M(f,I):\ m(I)\leqq r\}\ \text{for}\ 0<r\leqq 1,\ \text{and}\ M_0(f)=\lim_{r\to 0}M_r(f).$$

The quantity $M_1(f)$ is called the mean oscillation of f, and in case $M_1(f)<\infty$ we say that f has bounded mean oscillation (or that f is in BMO). In case $M_0(f)=0$ we say that f has vanishing mean oscillation (or that f is in VMO). The class BMO has recently been studied by Fefferman and Stein [7], who have proved, among other results, that a function belongs to BMO if and only if it can be written as $u+\tilde{v}$ where u and v belong to L^∞ and $\tilde{v}$ is the conjugate function of v. We need here the less difficult half of this equivalence, the half that asserts $L^\infty+(L^\infty)^\sim\subset$ BMO. This inclusion is bounded in the sense that there is a positive constant c with the property $M_1(u+\tilde{v})\leqq c(\|u\|_\infty+\|v\|_\infty)$ for all u, v in L^∞ [7].

The class VMO also has a simple characterization; it consists of all functions $u+\tilde{v}$ with u and v in C. Here we require only the inclusion $C+\tilde{C}\subset$ VMO, which can be proved as follows. As the inclusion $C\subset$ VMO is obvious, it will be enough to show that $\tilde{C}\subset$ VMO. Let v belong to C, and choose $\varepsilon>0$. Then there is a trigonometric polynomial p such that $\|v-p\|_\infty<\varepsilon$. Since $\tilde{p}$ is continuous we have

$$M_0(\tilde{v})=M_0(\tilde{v}-\tilde{p})\leqq M_1(\tilde{v}-\tilde{p})\leqq c\|v-p\|_\infty<c\varepsilon.$$

Because ε is arbitrary this shows that $M_0(\tilde{v})=0$, as desired.

Now it is trivial that a real L^∞ function belongs to $H^\infty+C$ if and only if it belongs to $C+\tilde{C}$. Thus, to guarantee that the f we construct satisfies condition (iii), it will suffice to arrange that $M_0(f)>0$.

To construct f we introduce the subarcs $I_n=\{e^{i\theta}:2^{-n}\leqq\theta\leqq 2^{-n}+5^{-n}\}$, $n=1,2,\dots$. We define f to be 0 off $\cup I_n$. On I_n we define f so that it is real, continuous, bounded in modulus by 1, vanishes at the endpoints of I_n, and satisfies

$$\int_{I_n} f\,dm=0,\quad \int_{I_n}|f|\,dm\geqq\frac{1}{2}m(I_n).$$

From the preceding equality and inequality we have $M_0(f) \geqq 1/2$, and thus (iii) holds. As (i) is obvious, it only remains to check (ii), which amounts to showing that the function $f(e^{i\theta})/\theta$ belongs to L^2. On I_n we have $|f/\theta| \leqq 2^n$, and so

$$\int_{I_n} |f/\theta|^2 \, dm \leqq 2^{2n} m(I_n) = (2\pi)^{-1} (4/5)^n.$$

The square-integrability of f/θ is now obvious, and the construction is complete.

It appears that any necessary and sufficient condition, in terms of the structures of f and g, for the compactness of $T_f T_g - T_{fg}$ will have to take account of subtleties of the behavior of the Gelfand transforms of f and g on the fibers of the Gelfand space of L^∞.

4. A partial converse

The above theorem does have a converse of sorts.

Theorem. *If g is in L^∞ and if $T_h T_g - T_{hg}$ is compact for all h in H^∞, then g is in $H^\infty + C$.*

This result was first conjectured by R. G. Douglas, who has independently found the following proof.

Under the hypotheses of the theorem, if h is any function in H^∞ and ψ is any inner function, then the operator

$$T_{\bar\psi}(T_h T_g - T_{hg}) = T_{\bar\psi h} T_g - T_{\bar\psi hg}$$

is compact. As the functions $\bar\psi h$ are dense in L^∞ [5], we may conclude that $T_f T_g - T_{fg}$ is compact for all f in L^∞.

For f in L^∞, let Γ_f be the Hankel operator induced by f, that is, the operator from H^2 to $(H^2)^\perp$ of multiplication by f followed by projection onto $(H^2)^\perp$. A theorem of Hartman [10] (see also [1]) states that Γ_f is compact if and only if f belongs to $H^\infty + C$. Now a simple calculation shows that $T_f T_g - T_{fg} = -\Gamma_{\bar f}^* \Gamma_g$, and thus $\Gamma_f^* \Gamma_g$ is compact for all f in L^∞. Taking $f = g$ we conclude that $\Gamma_g^* \Gamma_g$ is compact, and hence that Γ_g is compact. Therefore g is in $H^\infty + C$ by Hartman's theorem, as desired.

5. Proof of the lemma

We present here a simple direct proof of the lemma of Section 2. The proof depends on the fact that $H^\infty + C$ is an algebra [4].

Let f belong to L^∞. It is obvious that $\mathrm{dist}_\lambda(f, H^\infty) \leqq \mathrm{dist}(f, H^\infty + C)$ for each λ, so it will suffice to show that $\mathrm{dist}(f, H^\infty + C) \leqq \max\{\mathrm{dist}_\lambda(f, H^\infty) : |\lambda| = 1\}$. Let M denote the preceding maximum. Choose $\varepsilon > 0$, and for each λ choose an h_λ

in H^∞ such that dist $(f, h_\lambda) < M+\varepsilon$. Because $\mathrm{dist}_z(f, h_\lambda)$ is an upper semicontinuous function of z, there is for each λ an open subarc J_λ of the unit circle containing λ such that $\mathrm{dist}_z(f, h_\lambda) < M+2\varepsilon$ for all z in J_λ. Choose a finite number of the subarcs J_λ that cover the unit circle. Denote these subarcs by $J_1, \dots, J_p$ and the corresponding functions h_λ by $h_1, \dots, h_p$. Choose a partition of unity $\{w_k\}_{k=1}^q$ of the unit circle subordinate to the cover $\{J_n\}_{n=1}^p$ and consisting of nonnegative functions in C. Thus $\sum_1^q w_k = 1$ everywhere, and for each k there is an $n(k)$ such that $J_{n(k)}$ contains the support of w_k. By the latter property, if $w_k(\lambda) \neq 0$ then $\mathrm{dist}_\lambda(f, h_{n(k)}) < M+2\varepsilon$.

Now let $g = \sum_1^q w_k h_{n(k)}$. Then g is in $H^\infty + C$, and for any λ on the unit circle,

$$\mathrm{dist}_\lambda(f, g) = \mathrm{dist}_\lambda\Big(\sum_k w_k(\lambda) f, \sum_k w_k h_{n(k)}\Big) \leqq \sum_k \mathrm{dist}_\lambda\big(w_k(\lambda) f, w_k h_{n(k)}\big) =$$

$$= \sum_k \mathrm{dist}_\lambda\big(w_k(\lambda) f, w_k(\lambda) h_{n(k)}\big) = \sum_k w_k(\lambda)\, \mathrm{dist}_\lambda(f, h_{n(k)}) < \sum_k w_k(\lambda)(M+2\varepsilon) = M+2\varepsilon.$$

It follows that $\|f-g\|_\infty < M+2\varepsilon$. We may conclude that dist $(f, H^\infty + C) \leqq M$, and the lemma is proved.

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Über die Konvergenz von Reihen nach Produktsystemen

Von FERENC SCHIPP in Budapest

Herrn Professor B. Sz.-Nagy zum 60. Geburtstag gewidmet

Es sei (X, S, μ) ein Maßraum mit endlichem, positivem μ-Maß und $F=\{f_n: n\in N\}\subset L_\mu(X)$ ein Funktionensystem mit $M_n=\text{vrai}\max\limits_{x\in X}|f_n(x)|<\infty$ $(n\in N)$; $N=\{0, 1, 2, \ldots\}$. Es bezeichne $G=\{g_n: n\in N\}$ das Produktsystem des Funktionensystems F; das bedeutet, daß für $n\in N$ mit der dyadischen Darstellung $n=\sum\limits_{i=0}^{\infty} n_i 2^i$ $(n_i=0, 1)$

$$g_n=\prod_{i=0}^{\infty}(f_i/M_i)^{n_i} \tag{1}$$

gesetzt wird. Das System F heißt *stark multiplikativ*, wenn G ein Orthogonalsystem ist, *multiplikativ*, wenn $\int\limits_X g_n\,d\mu=0$ $(n\in N)$ ist; und *schwach multiplikativ*, wenn für sein Produktsystem G die Beziehung gilt:

$$\sum_{n=0}^{\infty}\left|\int_X g_n\,d\mu\right|=A<\infty. \tag{2}$$

Diese Begriffe wurden von G. ALEXITS [1], [2] eingeführt und in neuester Zeit von mehreren Mathematikern mit Erfolg auf wahrscheinlichkeitstheoretische Fragen angewandt.

Das Rademachersche System $R=\{r_n: n\in N\}$ ist ein stark multiplikatives System und das Produktsystem des Systems R ist das Walshsche Orthogonalsystem $W=\{w_n: n\in N\}$.

In [1] ist folgender Satz bewiesen (Theorème 2): Ist F ein gleichnormiertes stark multiplikatives System mit $M_n=1$ $(n\in N)$, so ist die Reihe $\sum c_n g_n$ unter der Bedingung $\sum c_n^2<\infty$ fast überall $(C, \alpha>0)$ summierbar.

In [3] wurde dieser Satz folgenderweise verschärft: Ist F ein schwach multiplikatives System und genügt sein Produktsystem G der Bedingung von I. SCHUR (s. [4], S. 70), so ist die Reihe $\sum c_n g_n$ im Falle $\sum c_n^2<\infty$ fast überall $(C, \alpha>0)$-summierbar.

Dieser Satz wird folgenderweise verallgemeinert:

Satz 1. *Ist F ein schwach multiplikatives System und G das Produktsystem von F, so konvergiert die Reihe* $\sum c_n g_n$ *unter der Bedingung* $\sum c_n^2 < \infty$ *fast überall.*

Es sei $f \in L^p[0, 1]$ $(1 \leqq p < \infty)$, $\hat{f}(n) = \int_0^1 f(t) w_n(t)\, dt$ $(n \in N)$, und für $x \in X$, $t \in [0, 1]$; $1 < s < \infty$ und $n = 1, 2, \dots$ führen wir die folgenden Bezeichnungen ein:

$$S_n(f)(t) = \sum_{k=0}^{n-1} \hat{f}(k) w_k(t), \quad T_n(f)(x) = \sum_{k=0}^{n-1} \hat{f}(k) g_k(x),$$

$$H_n^{(s)}(f)(t) = \left\{\frac{1}{n} \sum_{k=1}^{n} |S_k(f)(t)|^s\right\}^{1/s}, \quad K_n^{(s)}(f)(x) = \left\{\frac{1}{n} \sum_{k=1}^{n} |T_k(f)(x)|^s\right\}^{1/s}.$$

Wir werden beweisen:

Satz 2. *Ist* $f \in L^p[0, 1]$ $(1 < p < \infty)$, *so gilt für* $1 < s < \infty$ *(mit der Konstante A in* (2)*) und für* $n = 1, 2, \dots$:

$$\|K_n^{(s)}(f)\|_{X,p} \leqq \left(\frac{A-1}{p} + 1\right) \|H_n^{(s)}(f)\|_p.$$ [1]

Nach einem Satz von P. Sjölin ([5], Theorem C) gilt für $f \in L^p[0, 1]$ $(p > 1)$

$$\|\sup_n |S_n(f)|\|_p \leqq C_p \|f\|_p. \tag{3}$$

Da die Funktionen $H_n^{(s)}(f)(t)$, $K_n^{(s)}(f)(x)$ in bezug auf s monoton zunehmend sind und $\lim_{s\to\infty} H_n^{(s)}(f)(t) = \max_{1 \leqq k \leqq n} |S_k(f)(t)|$, $\lim_{s\to\infty} |K_n^{(s)}(x)| = \max_{1 \leqq k \leqq n} |T_k(f)(x)|$ ist, ergibt sich aus Satz 2 und aus (3) das folgende

Korollar. *Ist* $f \in L^p[0, 1]$ $(1 < p < \infty)$, *so gilt*

$$\|\sup_n |T_n(f)|\|_{X,p} \leqq B_p \|f\|_p. \tag{4}$$

Daraus folgt

Satz 3. *Ist* $f \in L^p[0, 1]$ $(p > 1)$, *so konvergiert die Reihe* $\sum_{n=0}^{\infty} \hat{f}(n) g_n$ *fast überall.*

Satz 1 ist offenbar in Satz 3 als Spezialfall enthalten.

Beweis des Satzes 2. Wir führen die „Kernfunktionen"

$$D_n(x, t) = \sum_{k=0}^{n-1} g_k(x) w_k(t) \qquad (t \in [0, 1];\ x \in X;\ n = 1, 2, \dots)$$

ein. Aus der Definition von G und W folgt

$$D_{2^k}(x, t) = \prod_{n=0}^{k-1} (1 + r_n(t) f_n(x)/M_n) \geqq 0 \quad (t \in [0, 1];\ x \in X),$$

[1] $\|_p$ und $\|_{X,p}$ bezeichnen Norm in $L^p[0, 1]$ und $L^p_\mu(X)$.

woraus sich nach (2)

$$(5)\qquad \int_X D_{2^k}(x,t)\,d\mu(x) \leqq A, \quad \int_0^1 D_{2^k}(x,t)\,dt = 1 \quad (t\in[0,1];\ x\in X;\ k\in N)$$

ergibt. Auf Grund der Orthogonalität des Walshschen Systems folgt

$$T_n(f)(x) = \sum_{m=0}^{n-1} \hat{f}(m) g_m(x) = \int_0^1 S_n(f)(t) D_{2^k}(x,t)\,dt \quad \text{für} \quad 2^k > n.$$

Es sei $1/p+1/p' = 1$ und $g\in L_\mu^{p'}(X)$ eine beliebige Funktion mit $\|g\|_{X,p'} \leqq 1$. Offenbar gibt es Funktionen $\alpha_m \in L_\mu(X)$ derart, daß mit $s' = s/(s-1)$ fast überall in X

$$K_n^{(s)}(f)(x) = \frac{1}{n^{1/s}} \sum_{m=1}^{n} \alpha_m(x) T_m(f)(x), \quad \left\{\sum_{m=1}^{n} |\alpha_m(x)|^{s'}\right\}^{1/s'} = 1$$

bestehen. Wir betrachten das lineare Funktional

$$I(g) = \int_X g K_n^{(s)}(f)\,d\mu(x) = \frac{1}{n^{1/s}} \int_X g \sum_{m=1}^{n} \alpha_m T_m(f)\,d\mu(x) =$$

$$= \frac{1}{n^{1/s}} \int_X \int_0^1 \left(\sum_{m=1}^{n} \alpha_m(x) S_m(f)(t)\right) g(x) D_{2^k}(x,t)\,dt\,d\mu(x).$$

Auf Grund der Hölderschen Ungleichung ergibt sich

$$|I(g)| \leqq \int_X \int_0^1 H_n^{(s)}(f)(t) \left\{\sum_{m=1}^{n} |\alpha_m(x)|^{s'}\right\}^{1/s'} g(x) D_{2^k}(x,t)\,dt\,d\mu(x) =$$

$$= \int_0^1 H_n^{(s)}(f)(t) \Big(\int_X g(x) D_{2^k}(x,t)\,d\mu(x)\Big)\,dt.$$

Daraus, durch Anwendung der Hölderschen Ungleichung folgt

$$I(g) \leqq \|H_n^{(s)}(f)\|_p \left\{\int_0^1 \Big|\int_X g(x) D_{2^k}(x,t)\,d\mu(x)\Big|^{p'}\right\}^{1/p'} = \|H_n^{(s)}(f)\|_p \cdot B.$$

Für beliebige $h\in L^p[0,1]$ mit $\|h\|_p \leqq 1$ sei

$$J(h) = \int_0^1 h(t) \Big(\int_X g(x) D_{2^k}(x,t)\,d\mu(x)\Big)\,dt.$$

Auf Grund der Ungleichung

$$|h(t) g(x)| \leqq \frac{|h(t)|^p}{p} + \frac{|g(x)|^{p'}}{p'}$$

und nach (5) ergibt sich

$$|J(h)| \leqq \frac{1}{p} \int_0^1 |h(t)|^p \Big(\int_X D_{2^k}(x, t)\, d\mu(x) \Big) dt +$$

$$+ \frac{1}{p'} \int_X |g(x)|^{p'} \Big(\int_0^1 D_{2^k}(x, t)\, dt \Big) d\mu(x) \leqq \frac{A}{p} + \frac{1}{p'}.$$

Daraus folgt, daß

$$B = \sup_{\|h\|_p \leqq 1} |J(h)| \leqq \frac{A}{p} + \frac{1}{p'}$$

gilt, und so ist

$$\|K_n^{(s)}(f)\|_{X,p} = \sup_{\|g\|_{X,p'}} |I(g)| \leqq \left(\frac{A}{p} + \frac{1}{p'} \right) \|H_n^{(s)}\|_p.$$

Damit ist Satz 2 bewiesen.

Beweis des Satzes 3. Aus (3) folgt, daß für $f \in L^p[0, 1]$ $(p > 1)$ $\lim_{n \to \infty} \|f - S_n(f)\|_p = 0.$ gilt. Aus der Folge $h_n = f - S_n(f)$ läßt sich eine Teilfolge $\{h_{n_k}\}$ mit $\sum_{k=1}^{\infty} \|h_{n_k}\|_p^p < \infty$ herausgreifen. Ist $n_k \leqq n < n_{k+1}$, so gilt

$$\varrho_n(x) = \sup_{m \geqq n} |T_m(f)(x) - T_n(f)(x)| \leqq 2 \sup_{m \geqq n_k} |T_m(f)(x) - T_{n_k}(f)(x)| =$$

$$= 2 \sup_m |T_m(h_{n_k})(x)| = \vartheta_k(x).$$

Da nach (4) $\|\vartheta_k\|_{X,p} \leqq B_p \|h_{n_k}\|_p$ gilt, so ist $\lim_{k \to \infty} \vartheta_k(x) = 0$ und gilt auch $\lim_{n \to \infty} \varrho_n(x) = 0$ fast überall.

Damit haben wir auch Satz 3 bewiesen.

Ich möchte Herrn Professor G. ALEXITS für seine wertvollen Ratschläge bei der Fertigstellung dieser Arbeit meinen aufrichtigen Dank aussprechen.

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A weakening of the definition of C*-algebras

By Z. SEBESTYÉN in Szeged

Dedicated to Professor Béla Sz.-Nagy on his 60th birthday

In a recent paper H. ARAKI and G. A. ELLIOTT proved the following theorem (see [1], Theorem 1):

Let A be a complex involutory algebra with complete linear space norm such that

(1) $$\|x^* \cdot x\| = \|x\|^2 \quad \textit{for all } x \in A.$$

A is then a C^*-algebra, i.e. the submultiplicativity property

(2) $$\|x \cdot y\| \leqq \|x\| \cdot \|y\|$$

also holds for every $x, y \in A$.

These authors raised the problem whether it is enough to assume (1) for normal x only, i.e. for which $x^*x = xx^*$.

The answer is in the negative as was shown in [5] by the simple counter example of the algebra A of all bounded linear operators on a complex Hilbert space with the numerical radius as norm. This norm does not satisfy

(3) $$\|x^*x\| \leqq \|x\|^2 \quad \text{for every} \quad x \in A.$$

The purpose of this note is to prove that (3), together with (1) only for normal $x \in A$, is sufficient for A to be a C^*-algebra.

We shall use the notation of RICKART's book [4]. The following lemma, similar to Lemma 1 in [5], plays an important role in the arguments. Denote by $H(A)$ the selfadjoint part of A.

Lemma 1. *Let A be a complex involutory algebra with linear space norm which satisfies (3). Then A is a normed algebra with continuous involution.*

Proof. The first step is to prove

(4) $$\|hk\| \leqq 4\|h\|\|k\| \quad \text{for every} \quad h, k \in H(A).$$

Consider for $h, k \in H(A)$ the identity

$$4hk = (h+k)^2 - (h-k)^2 + i(h+ik)(h-ik) - i(h-ik)(h+ik)$$

which is a special case of (3) in [1]. Use the triangle inequality together with (3) to have thus $\|hk\| \leqq (\|h\|+\|k\|)^2$. Assume that h and k differ from 0, otherwise (4) is immediate, and replace them by $h/\|h\|$ and $k/\|k\|$, respectively; then (4) follows immediately.

We define an auxiliary linear space norm as follows: for $h, k \in H(A)$ let

$$\|h+ik\|_1 = \frac{1}{\sqrt{2}} \sup \{\|h \cdot \cos t - k \cdot \sin t\| + \|h \cdot \sin t + k \cdot \cos t\| \colon t \text{ real number}\}$$

so that

$$\frac{1}{\sqrt{2}}(\|h\|+\|k\|) \leqq \|h+ik\|_1 \leqq \|h\|+\|k\|$$

holds (for details see [4], p. 7). Moreover, the 1-norm agrees with the original norm on $H(A)$ and the involution is an isometry with this norm. The multiplication is also continuous with the 1-norm as for all $x, y \in A$ the inequality

$$\|xy\|_1 \leqq 8\, \|x\|_1 \cdot \|y\|_1$$

holds. It follows that the norm of the extended left regular representation on A with 1-norm, defined for $x \in A$ by

$$\|x\|_2 = \sup \{\|\lambda x + xy\|_1 \colon \lambda \text{ complex number}, \quad y \in A; \quad |\lambda| + \|y\|_1 = 1\},$$

is an appropriate norm. Indeed, it is equivalent to the 1-norm, as it is not hard to see that

$$\|x\|_1 \leqq \|x\|_2 \leqq 8\, \|x\|_1$$

for any $x \in A$, so that the involution is a norm-continuous map with the 2-norm. This completes the proof.

In the following $\nu(x)$ denotes the spectral radius of $x \in A$ with respect to the 2-norm

$$\nu(x) = \lim \|x^n\|_2^{1/n}.$$

The next result is not an evident consequence of the Araki—Elliott theorem mentioned earlier, but it follows from Lemma 1 by the properties of the spectral radius.

Proposition 2. *Let A be a complex commutative involutory algebra with linear space norm such that* (1) *holds for any $x \in A$. Then A is a pre-C^*-algebra.*

Proof. We show first

$$\nu(h) = \|h\| \quad \text{for every} \quad h \in H(A) \tag{5}$$

by (1) and the equivalence of the norms on $H(A)$ as follows:

$$\nu(h) = \lim_{n\to\infty} \|h^{2^n}\|_2^{2^{-n}} = \lim_{n\to\infty} \|h^{2^n}\|_1^{2^{-n}} = \lim_{n\to\infty} \|h^{2^n}\|^{2^-} = \|h\|,$$

where in the last step the immediate consequence of (1)

$$\|h\ \ \| = \|h\|^{2^n} \qquad (n=1, 2, \ldots; \quad h \in H(A))$$

was used. The only required property of the original norm follows from (5) by (1):

$$\|xy\| = \|y^*x^*xy\|^{1/2} = \nu(x^*xy^*y)^{1/2} \leqq \nu(x^*x)^{1/2}\nu(y^*y)^{1/2} =$$
$$= \|x^*x\|^{1/2}\|y^*y\|^{1/2} = \|x\|\,\|y\|$$

holds for any $x, y \in A$. Thus A is a pre-C^*-algebra with the original norm in fact.
The main result of this paper is the following

Theorem 3. *Let A be a complex involutory algebra with complete linear space norm which satisfies* (3) *and for which* (1) *holds for every normal $x \in A$. Then A is a C^*-algebra.*

Proof. Proposition 2 implies that every maximal commutative selfadjoint subalgebra of A is a pre-C^*-algebra. Consider now $A^\sim$, the norm completion of A in the 2-norm with the extended involution. We shall show that $A^\sim$ is a C^*-algebra with an equivalent norm. In view of [2], Corollary 12 it suffices to prove that the set

$$\left\{\sum_{n=1}^{\infty}\frac{(i\tilde{h})^n}{n!} :\ \tilde{h}\in\tilde{A},\ \tilde{h}^* = \tilde{h}\right\}$$

is bounded in $A^\sim$. First for any normal $x \in A$ we have by a C^*-norm property

(6) $$\frac{1}{\sqrt{2}}\|x\| \leqq \|x\|_2 \leqq 8\left(\left\|\frac{x+x^*}{2}\right\| + \left\|\frac{x-x^*}{2}\right\|\right) \leqq 16\|x^*x\|^{1/2} = 16\|x\|$$

which gives for every $h \in H(A)$

(7) $$\frac{1}{\sqrt{2}}\left\|\sum_{n=1}^{\infty}(ih)^n/n!\right\| \leqq \left\|\sum_{n=1}^{\infty}(ih)^n/n!\right\|_2 \leqq 16\left\|\sum_{n=1}^{\infty}(ih)^n/n!\right\| \leqq 32,$$

since for the normal $\sum\limits_{n=1}^{\infty}(ih)^n/n! \in A^\sim$ the original norm can be extended by the previous equivalence and the quasi-unitary elements are of norms not greater than 2 in C^*-algebras. Let now a selfadjoint $h \in A$ and a positive number ε be given. Choose an $h \in H(A)$ which satisfies $\|h\|_2 \leqq \|\tilde{h}\|_2$ and $\|\tilde{h} - h\|_2 < \varepsilon \cdot e^{-\|\tilde{h}\|_2}$. Then (7) gives by a simple computation

$$\left\|\sum_{n=1}^{\infty}(i\tilde{h})^n/n!\right\|_2 \leqq \left\|\sum_{n=1}^{\infty}(ih)^n/n!\right\|_2 + \varepsilon \leqq 32 + \varepsilon,$$

where

$$\left\|\sum_{n=1}^{\infty}\frac{1}{n!}[(i\tilde{h})^n - (ih)^n]\right\|_2 \leqq \sum_{n=1}^{\infty}\frac{1}{n!}\left\|\sum_{m=0}^{n-1}(i\tilde{h})^{n-m}(ih)^m - (i\tilde{h})^{n-m-1}(ih)^{m+1}\right\|_2 =$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\left\|\sum_{m=0}^{n-1}(i\tilde{h})^{n-m-1}(i\tilde{h} - ih)(ih)^m\right\|_2 \leqq \|\tilde{h} - h\|_2 \sum_{n=1}^{\infty}\frac{\|\tilde{h}\|_2^{n-m-1}\|\tilde{h}\|_2^m}{(n-1)!} < \varepsilon$$

was used. This shows that 32 is an appropriate bound for the set considered abvoe. Thus A is a C^*-algebra with an equivalent norm, which agrees for every $x \in A$ with $\nu(x^*x)^{1/2}$ as well known from the C^*-condition. But thus (5) shows by the assumption for any $x \in A$

$$\nu(x^*x)^{1/2} = \|x^*x\|^{1/2} \leqq \|x\|. \tag{8}$$

We need show only the converse to (8) in the remainder. In case if A has an identity, for the C^*-norm we have by [3], (3. 7) Corollary the expression

$$\nu(x^*x)^{1/2} = \inf\left\{\sum_{j=1}^{n} |\lambda_j| : x = \sum_{j=1}^{n} \lambda_j \exp(i\tilde{h}_j),\ \tilde{h}_j = \tilde{h}_j^* \in \tilde{A};\ n = 1, 2, \dots\right\}. \tag{9}$$

Assuming now $\sum_{j=1}^{n} |\lambda_j| < \nu(x^*x)^{1/2} + \varepsilon/2$ for some $\varepsilon > 0$ such that $x = \sum_{j=1}^{n} \lambda_j \exp(i\tilde{h}_j)$ holds with $\tilde{h}_j \in A^{\sim}$, $\tilde{h}_j^* = \tilde{h}_j$ $(j = 1, 2, \dots, n)$, we can choose normal $x_j \in A$ which satisfy $\|x_j\| = 1$, $\|\exp(i\tilde{h}_j) - x_j\|_2 < \varepsilon/2\sqrt{2} \sum_{j=1}^{n} |\lambda_j|$ for $j = 1, 2, \dots, n$. Then using (6) we have

$$\|x\| \leqq \left\|x - \sum_{j=1}^{n} \lambda_j x_j\right\| + \left\|\sum_{j=1}^{n} \lambda_j x_j\right\| < \sqrt{2} \sum_{j=1}^{n} |\lambda_j| \, \|\exp(i\tilde{h}_j) - x_j\|_2 + \sum_{j=1}^{n} |\lambda_j| <$$
$$< \nu(x^*x)^{1/2} + \varepsilon.$$

Since ε was an arbitrary positive number, the converse to (8) is valid as claimed. Suppose finally that A has not an identity. Then analogously

$$\nu(x^*x)^{1/2} = \inf\left\{\sum_{j=1}^{n} |\lambda_j| : x = \sum_{j=1}^{n} \lambda_j \sum_{m=1}^{\infty} (i\tilde{h}_j)^m/m! :\ \tilde{h}_j = \tilde{h}_j^* \in \tilde{A},\ (j = 1, 2, \dots, n),\ n = 1, 2, \dots\right\}$$

holds where $\sum_{j=1}^{n} \lambda_j = 0$ is automatically satisfied. The proof of the converse to (8) can be done in an analogous way. The proof of the theorem is complete.

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On a Fourier $L^1(E_n)$-multiplier criterion

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Dedicated to Professor B. Sz.-Nagy on the occasion of his 60th birthday

The main purpose of this note is to give a simple sufficient criterion for radial functions on the Euclidean n-space E_n to be the Fourier transform of an integrable function. The present criterion is a partial generalization of a well-known one-dimensional result due to Sz.-NAGY [9], namely

Theorem A. Let $h(v)$ be an even (continuous) function on $(-\infty, \infty)$ satisfying the following conditions i) $h(v) \to 0$ for $v \to \infty$, ii) $h'(v) \in L(0, \infty)$, iii) h' is locally of bounded variation except at the points $a_0 = 0 < a_1 < \cdots < a_s < \infty$ but in the neighbourhoods of a_i the integrals $\int_{0+} v\,|dh'(v)|$,

$$\Big(\int^{a_i -} + \int_{a_i +}\Big) |v - a_i| \log (1/|v - a_i|)\,|dh'(v)| \quad (1 \leqq i \leqq s < \infty),$$

and $\int^{\infty} v\,|dh'(v)|$ converge. Then there exists an even integrable function H such that

$$h(v) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} H(x) e^{-ivx}\, dx.$$

The case $s = 0$ is the one considered in [1], [2]; for further details see also [4; p. 251, p. 276].

Apart from a regularization at the critical points a_i, $1 \leqq i \leqq s$, introduced in the course of a partial integration, the proof mainly depends upon the absolute integrability of the Fejér-kernel on $(-\infty, \infty)$. For the n-dimensional analogue we will make heavy use of the absolute integrability of a suitable Riesz-kernel. This paper was written while the author held a DFG-fellowship; the author thanks Professor R. J. NESSEL for a careful reading of the manuscript.

First let us give some notations. Let v, x, y denote elements of $E_n (x = (x_1, \ldots, x_n))$, $x \cdot y = \sum_{k=1}^{n} x_k y_k$ the inner product, $|x| = x \cdot x^{1/2}$ the absolute value; let $m = (m_1, \ldots, m_n)$

be an n-tuple of non-negative integers with $|m| = \sum_{k=1}^{n} m_k$, D^m the differential operator $(\partial/\partial x_1)^{m_1} \ldots (\partial/\partial x_n)^{m_n}$, $[\alpha]$ the largest integer less than or equal to $\alpha \in E_1$. A function $f(x)$, defined on E_n, is called radial if $f(x) = f(|x|)$. Let the Fourier transformation on $\mathbf{L}^1(E_n)$, the set of all integrable functions on E_n, be defined by

$$[f]^\wedge(v) \equiv f^\wedge(v) = (2\pi)^{-n/2} \int_{E_n} f(x) e^{-iv \cdot x}\, dx$$

and let $[\mathbf{L}^1(E_n)]^\wedge$ be the set of all continuous functions which are equal to the Fourier transform of an $\mathbf{L}^1$-function.

For our multiplier theorem it is convenient to introduce the class $\mathbf{BV}_{j+1}$ consisting of those continuous functions h on $[0, \infty)$ such that $h, \ldots, h^{(j-2)}$ are absolutely continuous on $(0, \infty)$, $h^{(j-1)}$ locally absolutely continuous, $\lim_{\tau \to \infty} h^{(i)}(\tau) = 0$ for $0 \leqq i \leqq j-1$, and $h^{(j)}$ locally of bounded variation on $(0, \infty)$ with

$$\int_0^\infty \tau^j |dh^{(j)}(\tau)| < \infty. \tag{1}$$

It follows readily that

$$\mathbf{BV}_{j+1} \subset \mathbf{BV}_j. \tag{2}$$

Indeed, for ε, $R > 0$ and $h \in \mathbf{BV}_{j+1}$ one has

$$\int_\varepsilon^R \tau^j\, dh^{(j)}(\tau) = \sum_{\nu=0}^{j} (-1)^\nu \frac{j!}{(j-\nu)!} \tau^{j-\nu} h^{(j-\nu)}(\tau)\Big|_\varepsilon^R$$

which, by hypothesis, remains bounded for $R \to \infty$. Observing that $\lim_{R \to \infty} h^{(j-\nu)}(R) = 0$, $1 \leqq \nu \leqq j$, one necessarily has $h^{(j)}(R) \to 0$ for $R \to \infty$. Now Dirichlet's formula yields

$$\int_0^\infty \tau^{j-1} |h^{(j)}(\tau)|\, d\tau = \int_0^\infty \tau^{j-1} \Big| \int_\tau^\infty dh^{(j)}(\omega) \Big|\, d\tau \leqq$$

$$\leqq \int_0^\infty |dh^{(j)}(\omega)| \int_0^\omega \tau^{j-1}\, d\tau = j^{-1} \int_0^\infty \omega^j |dh^{(j)}(\omega)|.$$

The classes $\mathbf{BV}_{j+1}$ have already been considered in BUTZER—NESSEL—TREBELS [5] in order to obtain a simple estimate of

$$\sum_{k=0}^{\infty} \binom{k+j}{j} |\Delta^{j+1} \alpha_k| < \infty, \quad \Delta \alpha_k = \alpha_k - \alpha_{k+1}, \quad \Delta^{j+1} = \Delta \Delta^j, \tag{3}$$

the latter being a multiplier condition on a Banach space with a total sequence $\{P_k\}$ of orthogonal bounded linear projections under the hypothesis that $f \sim \sum P_k f$ is (C, j)-bounded. In this respect, the following theorem is the concrete continuous analogue of the abstract discrete multiplier theorem mentioned above. Indeed it is quite natural to replace the (C, j)-boundedness of the abstract Fourier expansion by the boundedness of the corresponding $(1, j)$-Riesz-means in case of Fourier integrals. Here the $(\varkappa, \lambda)$-Riesz-means are defined for $\varkappa, \lambda > 0$ on $\mathbf{S}$ (the set of infinitely differentiable, rapidly decreasing functions) by

$$R_{\varkappa,\lambda}(\varrho)f = \varrho^n r_{\varkappa,\lambda}(\varrho\,\cdot) * f, \quad [r_{\varkappa,\lambda}]^\wedge(v) = \begin{cases} (1-|v|^\varkappa)^\lambda, & |v| \leqq 1 \\ 0, & |v| \geqq 1 \end{cases} \tag{4}$$

where $*$ convolution, $r_{\varkappa,\lambda}$, and its Fourier transform are to be understood in the distributional sense. It is known (see e.g. [6]) that

$$r_{\varkappa,\lambda} \in \mathbf{L}^1(E_n) \quad \text{for} \quad \varkappa > 0, \quad \lambda > (n-1)/2; \tag{5}$$

thus, (4) is meaningful for all $f \in \mathbf{L}^1(E_n)$ for these $\varkappa$, λ-values and $[r_{\varkappa,\lambda}]^\wedge$ exists in the classical sense.

Theorem 1. *If* $h \in \mathbf{BV}_{j+1}$ *for* $j = [(n-1)/2]+1$, *then* $h(|v|) \in [\mathbf{L}^1(E_n)]^\wedge$.

Proof. Consider the function

$$H(x) = [(-1)^j/j!] \int_0^\infty \tau^{j+n} r_{1,j}(\tau x)\, dh^{(j)}(\tau)$$

which is integrable on account of the hypothesis and (5):

$$\int_{E_n} |H(x)|\, dx \leqq \int_0^\infty \tau^j |dh^{(j)}(\tau)| \int_{E_n} \tau^n |r_{1,j}(\tau x)|\, dx < \infty.$$

Passing to Fourier transforms, by Fubini's theorem and partial integration

$$H^\wedge(v) = ((-1)^j/j!) \int_0^\infty \tau^j \begin{Bmatrix} \left(1-\dfrac{|v|}{\tau}\right)^j, & |v| \leqq \tau \\ 0, & |v| \geqq \tau \end{Bmatrix} dh^{(j)}(\tau) =$$

$$= ((-1)^j/j!) \int_{|v|}^\infty (\tau-|v|)^j\, dh^{(j)}(\tau) =$$

$$= ((-1)^j/(j!)) \left\{ (\tau-|v|)^j h^{(j)}(\tau) \Big|_{|v|}^\infty - j \int_{|v|}^\infty (\tau-|v|)^{j-1} h^{(j)}(\tau)\, d\tau \right\}.$$

Now $h^{(j)}(\tau)$ is locally of bounded variation in $(0, \infty)$, and therefore the first term vanishes at $\tau = |v|$. Since $h^{(j)}(\tau) \to 0$ for $\tau \to \infty$, it follows that

$$|R^j h^{(j)}(R)| = \left| R^j \int_R^\infty dh^{(j)}(\tau) \right| \leqq \int_R^\infty \tau^j |dh^{(j)}(\tau)|$$

becomes infinitely small for $R\to\infty$. Hence

$$H^\wedge(v) = ((-1)^{j-1}/(j-1)!)\int_{|v|}^{\infty}(\tau-|v|)^{j-1}h^{(j)}(\tau)\,d\tau$$

and thus, proceeding iteratively,

$$H^\wedge(v) = -\int_{|v|}^{\infty}(\tau-|v|)h^{(2)}(\tau)\,d\tau = \int_{|v|}^{\infty}h'(\tau)\,d\tau = h(|v|).$$

Using (5) for arbitrary $\varkappa>0$, $\lambda=j=[(n-1)/2]+1$, it is clear by the above proof that $|v|$ may be replaced by $|v|^{\varkappa}$ provided $\int_0^{\infty}|v|^{\varkappa j}|dh^{(j)}(|v|^{\varkappa})|<\infty$ and $\lim_{\tau\to\infty}h^{(i)}(\tau)=0$ $(0\leqq i\leqq j-1)$ which, however, is equivalent to $h\in\mathbf{BV}_{j+1}$ on account of the homogeneity of the integral. Thus

Corollary. *If* $h\in\mathbf{BV}_{j+1}$ *for* $j=[(n-1)/2]+1$, *then* $h(|v|^{\varkappa})\in\mathbf{L}^1[(E_n)]^\wedge$ *for* $\varkappa>0$.

Obviously, Theorem 1 is a generalization of Sz.-Nagy's theorem in case $i=0$ to n-dimensions. In case there is a further singularity at the point $a\in(0,\infty)$ one could proceed analogously as in Sz.-Nagy's proof provided one can estimate $\{a^{j+n}r_{1,j}(ax)-\tau^{j+n}r_{1,j}(\tau x)\}$ in the $\mathbf{L}^1(E_n)$-norm conveniently, e.g. by $O(|a-\tau|^{\alpha})$ for some $\alpha>0$. However, we do not pursue this aspect further since there exists a convenient general multiplier theorem of Löfström [6, 7] dealing with such a finite number of singularities.

In case that singularities are admitted only at the origin and/or at infinity, Löfström's result was improved by Boman [3] to ($\mathbf{C}^N(A)$ the set of all N-times continuously differentiable functions on the open $A\subset E_n$):

Theorem B. a) If $f\in\mathbf{C}^N(E_n)$, where $N=[n/2]+1$, and there exist constants C and $\delta>0$ such that

$$|D^m f(x)|\leqq C|x|^{-\delta-|m|}\quad (x\in E_n,\quad 0\leqq|m|\leqq N),$$

then $f\in[\mathbf{L}^1(E_n)]^\wedge$.

b) Let $f\in\mathbf{C}^N(E_n\setminus\{0\})$, $N=[n/2]+1$, have compact support, and let there exist constants C and $\delta>0$ such that

$$|D^m f(x)|\leqq C|x|^{\delta-|m|}\quad (x\in E_n\setminus\{0\},\quad 0\leqq|m|\leqq N),$$

then $f\in[\mathbf{L}^1(E_n)]^\wedge$.

To illustrate the range of Theorems 1 and B, consider

$$f_1(x) = \{1+\log(1+|x|^2)\}^{-1}.$$

Obviously, f_1 is radial and belongs to $\mathbf{C}^\infty(E_n)$; but since f_1 decreases too weakly at infinity, Theorem B does not apply immediately, whereas a simple calculation shows that $f_1 \in \mathbf{BV}_{j+1}$. Thus $f_1 \in [\mathbf{L}^1(E_n)]^\wedge$ by Theorem 1. Analogously one has (cf. Corollary) $(1+\log\log(e+|x|^\varkappa))^{-\alpha} \in [\mathbf{L}^1(E_n)]^\wedge$ for $\varkappa, \alpha>0$, etc.

To give an example with a singularity at the origin choose $f_2(x) = -\log^{-1}|x|\chi(|x|)$ with some $\chi \in \mathbf{C}^\infty(E_n)$ satisfying $\chi(x)=1$ for $0 \leqq |x| \leqq 1/e$ and $=0$ for $|x| \geqq 2/e$. Again, Theorem 1 yields $f_2 \in [\mathbf{L}^1(E_n)]^\wedge$, whereas Theorem B does not apply.

Naturally one could try "Bernstein's multiplier theorem" (see PEETRE [8]): $\dot{\mathbf{W}}^{n/2,1} \subset [\mathbf{L}^1(E_n)]^\wedge$, where $\dot{\mathbf{W}}^{n/2,1}$ may be equivalently characterized by

$$\int_0^\infty \tau^{-n/2} \sup_{|y| \leqq \tau} \|\Delta_y^n f(x)\|_2 \frac{d\tau}{\tau} < \infty$$

with $\Delta_y f(x) = f(x+y)-f(x)$. But to verify this condition in case of the above examples seems to be far harder than to check that $f \in \mathbf{BV}_{j+1}$ (other characterizations of $\dot{\mathbf{W}}^{n/2,1}$, known to the author, seem to be still more complicated).

The obvious disadvantage of Theorem 1 lies in the assumption that f has to be radial. Here another criterion, overlapping with Theorem 1 and but in some examples stronger than Theorem B, may help. Its proof rests upon the integrability of the Riesz-kernel $r_{\varkappa,1}$ on E_1 for $\varkappa>0$, so that the product kernel $\prod_{k=1}^n r_{\varkappa_k,1}(x_k)$ is integrable on E_n. Thus

Theorem 2. *Let f be a continuous function on E_n, even in each coordinate, differentiable in the sense that for $0 \leqq m_i \leqq 2$ the derivatives $D^m f(x)$ exist as locally integrable functions, that $\lim_{x_k \to \infty} D^m f(x)=0$ for $m_k=0$ or 1 or 2, $1 \leqq k \leqq n$ and that*

$$\int_0^\infty \dots \int_0^\infty |D^m f(x)| \prod_{m_k \neq 0} x_k^{m_k-1} dx_k < \infty$$

uniformly in x_k when $m_k=0$. Then $f(|x_1|^{\varkappa_1}, \dots, |x_n|^{\varkappa_n}) \in [\mathbf{L}^1(E_n)]^\wedge$ provided $\varkappa_k>0$, $1 \leqq k \leqq n$.

For the proof consider

$$F(y) = \int_0^\infty \dots \int_0^\infty \prod_{k=1}^n x_k^{1+(1/\varkappa_k)} r_{\varkappa_k}(x_k^{1/\varkappa_k} y_k) f^{(2n)}_{x_1 x_1 \dots x_n x_n}(x)\, dx,$$

which is clearly integrable, and proceed as in the proof of Theorem 1.

Theorem 2 is another generalization of Sz.-Nagy's theorem [9]; in case $\varkappa_k=1$, $1 \leqq k \leqq n$, his estimate of $\{a^2 r_{1,1}(a\eta) - \tau^2 r_{1,1}(\tau\eta)\}$, $a, \eta \in (0, \infty)$ in the $\mathbf{L}^1(-\infty, \infty)$-norm may be taken over to cover singularities on the hyperplanes $x_k=a>0$, $1 \leqq k \leqq n$. Thus *a* theorem may be stated which is analogous to Theorem A. But instead of formulating it, let us give an example to which Theorem 2 applies but Theorem B does not since i) the function decreases at infinity too slowly, ii) Theorem B allows only a singularity at the origin and not on the hyperplanes $x_k=0$, $1 \leqq k \leqq n$. It is

$$\left(1+\log\left(1+|x_1|^{\varkappa_1}+\cdots+|x_n|^{\varkappa_n}\right)\right)^{-n} \in [\mathbf{L}^1(E_n)]^{\wedge}$$

provided $\varkappa_k>0$, $1 \leqq k \leqq n$, as can easily be shown by Theorem 2.

Let us conclude with the remark that Theorems 1 and 2 are based upon summability properties of the Fourier integral in direct analogy to the abstract series case as elaborated in [5].

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Spectra of finite range Cesàro operators

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In [BHS] BROWN, HALMOS, and SHIELDS studied the operators C_0, C_1, C_∞ defined respectively on the spaces l^2, $L^2(0, 1)$, $L^2(0, \infty)$, by

$$C_0x(n) = \frac{1}{n+1}\sum_{k=0}^{n} x(k), \quad C_1x(t)=\frac{1}{t}\int_0^t x(s)\,ds, \quad C_\infty x(t)=\frac{1}{t}\int_0^t x(s)\,ds.$$

In particular they determined, using Hilbert space techniques, that the adjoint of $I-C_1$ is a simple unilateral shift and the adjoint of $I-C_\infty$ is a simple bilateral shift, from which it follows that the spectrum of C_1 is the disk $\{\lambda : |1-\lambda| \leqq 1\}$ (with point spectrum the open disk $\{\lambda : |1-\lambda| < 1\}$) and the spectrum of C_∞ is the circle $\{\lambda : |1-\lambda| = 1\}$ (with point spectrum empty). We should point out that the fact that $I-C_\infty^*$ is unitary and has spectrum the unit circle can be obtained in another way, following ideas of GOLDBERG [G]. After mapping $L^2(0, \infty)$ isometrically onto the L^2 space of the multiplicative group G of positive real numbers (with respect to the Haar measure $\frac{dt}{t}$) via $x(t) \to t^{\frac{1}{2}}x(t)$, we see that C_∞^*, which is given by $C_\infty^* x(t) = \int_t^\infty \frac{x(s)}{s}\,ds$, is the operation of convolution by a certain function $\varphi \in L^1(G)$. Using the usual notation for Fourier transforms, one computes directly that $1-\hat{\varphi}$ has modulus identically 1 and has each point of the unit circle in its essential range. It follows at once that $I-C_\infty^*$ is unitarily equivalent to an operator which is unitary and has the entire unit circle as its spectrum.

In [Bo], BOYD used an explicit integral formula for the resolvent to show that the corresponding operator T_∞ on $L^p(0, \infty)$ is a bounded operator mapping $L^p(0, \infty)$ into itself and having spectrum the circle $\left\{\lambda : \operatorname{Re}\frac{1}{\lambda} = \frac{p-1}{p}\right\}$ for $1<p\leqq\infty$ (with $\frac{p-1}{p}$ defined to be 1 if $p=\infty$).

Here we determine the spectrum of the corresponding operator T_1 on the space $L^p(0, 1)$ $(1<p\leqq\infty)$ and add a few remarks concerning T_∞.

Theorem. *Let* $1<p<\infty$ *and let* $(T_1x)(t)=t^{-1}\int_0^t x(s)ds$ for $x\in L^p(0, 1)$. *Then* T_1 *is a bounded linear operator on* $L^p(0, 1)$. *The spectrum of* T_1 *is the closed disk* $D_p=\left\{\lambda\colon \mathrm{Re}\frac{1}{\lambda}\geqq\frac{p-1}{p}\right\}$. *Each eigenvalue of* T_1 *has multiplicity* 1, *and the point spectrum of* T_1 *is the interior of* D_p.

Proof. By Hardy's inequality for integrals [HLP, p. 240], if $y\in L^p(0, \infty)$ then $T_\infty y\in L^p(0, \infty)$ and $\|T_\infty y\|_p<\frac{p}{p-1}\|y\|_p$ unless $y=0$ a.e. Hence T_∞ is a bounded operator on $L^p(0, \infty)$, and since the constant is best possible, $\|T_\infty\|_p=\frac{p}{p-1}$. From this it follows that T_1 is a bounded operator on $L^p(0, 1)$ with norm at most $\frac{p}{p-1}$. For if $x\in L^p(0, 1)$ and $\tilde{x}(t)=x(t)\,(0<t<1)$, $\tilde{x}(t)=0\,(t\geqq 1)$, then

$$\|T_1x\|_p=\left(\int_0^1 |T_\infty\tilde{x}(t)|^p\,dt\right)^{1/p}\leqq\|T_\infty\tilde{x}\|_p\leqq\frac{p}{p-1}\|\tilde{x}\|_p=\frac{p}{p-1}\|x\|_p.$$

We observe that if $x\in L^p(0, 1)$, then $x\in L^1(0, 1)$ and hence T_1x is a continuous function on $(0, 1)$. In particular, the range of T_1 is a proper subspace of $L^p(0, 1)$ so 0 belongs to the spectrum of T_1.

If $\lambda\neq 0$ and $T_1x=\lambda x$, it follows that x is continuous and hence by the fundamental theorem of calculus, that x is differentiable. Differentiating the relation $\lambda tx(t)=$ $=\int_0^t x(s)ds$, we have

$$\lambda tx'(t)+(\lambda-1)x(t)=0.$$

This is an Euler differential equation of first order and thus its solutions have the form $x(t)=ct^\alpha$ where α is a complex scalar. We find $\lambda\alpha+(\lambda-1)=0$ or $\alpha=\frac{1}{\lambda}-1$. (Thus, considered as a mapping from the space of integrable functions to the space of continuous functions on $(0, 1)$, T_1 has every nonzero number as a simple eigenvalue.) Since $t^\alpha\in L^p(0, 1)$ iff $\mathrm{Re}\,(\alpha p)>-1$, $t^{\frac{1}{\lambda}-1}\in L^p(0,1)$ iff $\mathrm{Re}\frac{1}{\lambda}>\frac{p-1}{p}$. So the point spectrum of T_1 is the interior of D_p and every eigenvalue of T_1 has geometric multiplicity 1. (Moreover, since $t^\alpha\notin L^p(0, \infty)$ for any α, the operator T_∞ has a void point spectrum.)

Next let the transformations P_ζ be defined by

$$(P_\zeta x)(t) = \int_0^1 s^{-\zeta} x(st)\, ds.$$

Then by Boyd's formula, P_ζ is a bounded operator on $L^p(0, 1)$ if $\operatorname{Re} \zeta < \frac{p-1}{p}$ and for such ζ, $\zeta P_\zeta T_1 = \zeta T_1 P_\zeta = P_\zeta - T_1$. Hence if $\operatorname{Re} \frac{1}{\lambda} < \frac{p-1}{p}$ and $\zeta = \lambda^{-1}$, we see that $-\zeta^2 P_\zeta - \zeta I$ is a bounded operator inverse for $T_1 - \lambda I$; so λ belongs to the resolvent set for T_1. Thus $\sigma(T_1) \subset D_p$.

The spectrum of a bounded operator being compact, we must have $\sigma(T_1) = D_p$. We observe that the condition $\operatorname{Re} \frac{1}{\lambda} \geqq \frac{p-1}{p}$ is equivalent to the condition: $|\lambda|^2 \leqq \frac{p-1}{p} \operatorname{Re} \lambda$, so that D_p is the disk with center $\left(\frac{q}{2}, 0\right)$ and radius $\frac{q}{2}$ in R^2 (where $q = \frac{p-1}{p}$ is the conjugate index to p). Q.E.D.

The argument needs a slight modification when $p = \infty$. Since $t^\alpha \in L^\infty(0, 1)$ iff $\operatorname{Re} \lambda \geqq 0$, we find that $t^{\frac{1}{\lambda} - 1}$ is an eigenvector of T_1 on $L^\infty(0, 1)$ corresponding to the eigenvalue λ iff $\operatorname{Re} \frac{1}{\lambda} \geqq 1$. (Since $t^\alpha \in L^\infty(0, \infty)$ iff $\operatorname{Re} \alpha = 0$, the eigenvalues of T_∞ acting on $L^\infty(0, \infty)$ are the scalars λ with $\operatorname{Re} \frac{1}{\lambda} = 1$. Hence the operator T_∞ on $L^\infty(0, \infty)$ has spectrum which is entirely point spectrum.) Boyd's formula is still applicable, so $T_1 - \lambda I$ is invertible if $\operatorname{Re} \frac{1}{\lambda} < 1$. We summarize as follows.

Theorem. *Let T_1 be defined by the formula above. Then T_1 is a bounded linear operator on $L^\infty(0, 1)$. The spectrum of T_1 is the closed disk $D_\infty = \left\{\lambda : \operatorname{Re} \frac{1}{\lambda} \geqq 1\right\}$. The point spectrum of T_1 is $D_\infty \setminus \{0\}$ and each eigenvalue of T_1 has multiplicity* 1.

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Spectra of convolution operators

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1. Introduction. A number of recent papers have dealt with the question of determining the spectrum of operators which are special cases of the following type of operator:

$$Tf(t) = \int_0^\infty k(s) f(ts)\, ds. \tag{1}$$

Here k is a given measurable function and the operator is considered as a mapping from $L^p(0, \infty)$ into itself. A sufficient condition for T to act as a bounded operator from $L^p(0, \infty)$ to itself is the well known result of HARDY, LITTLEWOOD and PÓLYA [6, p. 230] to the effect that

$$\|T\|_p \leqq \int_0^\infty |k(s)|\, s^{-1/p}\, ds = N_p(k) < \infty. \tag{2}$$

For example, BROWN, HALMOS and SHIELDS [2] by Hilbert space methods found the spectrum of the Cesàro operator

$$Pf(t) = \frac{1}{t} \int_0^t f(s)\, ds = \int_0^1 f(ts)\, ds. \tag{3}$$

In [1], this author gave an explicit formula for the resolvent of P as an operator on $L^p(0, \infty)$, $1 \leqq p \leqq \infty$, and from this deduced the spectrum of P. LEIBOWITZ [10] determined the spectrum of P as an operator on $L^p[0, 1]$. Recently, RHOADES [11] extended the considerations to operators corresponding to Gamma type summation methods. LEIBOWITZ [9] has determined the spectrum of operators of the type (1) where $k(s)$ vanishes for $s \geqq 1$, and for some $\varepsilon > 0$ satisfies

$$\int_0^1 k(s)\, s^{\varepsilon - (1/p)}\, ds < \infty. \tag{4}$$

Rhoades and Leibowitz also consider these operators as acting on $L^p(0, 1)$, and

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Leibowitz completely determines the spectrum in this case without the extra condition (4).

It is well known (see for example [12, p. 304, p. 311], [13, p. 36]) that operators of the type (1) are essentially convolution operators. This fact was used in [9]. Using this, it is clear that the spectrum of T as an operator on $L^p(0, \infty)$ is exactly the spectrum of the following operator $K*$ acting on $L^p(\mathbf{R})$:

$$K*u(x) = \int_{-\infty}^{\infty} K(x-y)\,u(y)\,dy, \tag{5}$$

where $K(x)=k(e^{-x})e^{-x/q}$, $(q = p/(p-1))$. The condition (2) translates into the condition $\|K*\|_p \leqq \|K\|_1$ which is a familiar inequality for convolutions [3, p. 528], [12, p. 97]. Note that the expression $\|K*\|_p$ denotes the operator norm of $K*$ acting on $L^p(\mathbf{R})$.

It is surely a familiar fact that the spectrum of $K*$ acting on $L^p(\mathbf{R})$ is the closure of the range of $\hat{K}$, the Fourier transform of K. Since we have been unable to locate a proof of this in the literature except for $p=1$ and 2, a proof is presented here as Theorem 1. From this it follows that the spectrum of T in $L^p(0, \infty)$ is the closure of the range of the Mellin transform

$$\hat{k}\left(-\frac{1}{p}+i\xi\right) = \int_0^{\infty} k(s)\,s^{-(1/p)+i\xi}\,ds. \tag{6}$$

For completeness, we also present some results concerning the point spectrum of convolution operators (Theorem 2) and point out that the Riesz—Thorin theorem produces an interesting inequality when applied to operators of type (1).

2. Convolution operators. In this section we will consider the operator $K*$ defined by (5) for $K \in L^1(\mathbf{R})$. We denote the Fourier transform of K by

$$\hat{K}(\xi) = \int_{-\infty}^{\infty} K(x)\,e^{i\xi x}\,dx. \tag{7}$$

We will always assume that $1 \leqq p \leqq \infty$. The spectrum of a bounded operator from a Banach space X into itself will be denoted by $\sigma(T; X)$.

The following deep result is due to WIENER and now usually established within the framework of the theory of Banach Algebras. See [4, p. 107] for a proof.

Lemma 1. *Let $K \in L^1(\mathbf{R})$ and suppose that λ is a complex number such that $\lambda \neq 0$, and $\lambda \neq \hat{K}(\xi)$ for any $\xi \in \mathbf{R}$. Then there is a function $A_\lambda \in L^1(\mathbf{R})$ such that*

$$\lambda A_\lambda - K * A_\lambda = K. \tag{8}$$

Corollary 1. *The spectrum of $K*$ as an operator on $L^p(\mathbf{R})$ is contained in the closure of the range of $\hat{K}$ on $\mathbf{R}$.*

Proof. If λ is not in the given set then by Lemma 1, there is an $A_\lambda \in L^1(\mathbf{R})$ satisfying (8). Since convolution is a commutative operation, one readily verifies that the operator $\lambda^{-1}(I+A_\lambda *)$, which is a bounded operator on $L^p(\mathbf{R})$, is the inverse of $(\lambda - K*)$, so λ is in the resolvent set of $K*$.

Lemma 2. *Let $1<p<\infty$. Let $K \in L^1(\mathbf{R})$ and suppose that*

$$(9) \qquad \int_{-\infty}^{\infty} |xK(x)|\,dx = M < \infty.$$

Then, for each $\xi \in \mathbf{R}$, and $\delta > 0$, there are functions $u_\delta \in L^p(\mathbf{R})$ of unit norm such that

$$\|\hat{K}(\xi)\,u_\delta - K * u_\delta\|_p = 0(\delta) \quad \text{as} \quad \delta \to 0.$$

Proof. For any $\delta > 0$ and $\xi \in \mathbf{R}$, let

$$(10) \qquad v_\delta(x) = \int_{\xi-\delta}^{\xi+\delta} e^{-i\eta x}\,d\eta = 2e^{-i\xi x}(\sin \delta x)/x.$$

Then $v_\delta \in L^p(\mathbf{R})$ for $1<p<\infty$ and

$$(11) \qquad \|v_\delta\|_p = \delta^{1-(1/p)}\|v_1\|_p.$$

Also, by interchange of order of integration, we have

$$(12) \qquad K * v_\delta(x) = \int_{\xi-\delta}^{\xi+\delta} e^{-i\eta x}\hat{K}(\eta)\,d\eta.$$

Thus

$$(13) \qquad E_\delta(x) = \hat{K}(\xi)\,v_\delta(x) - K * v_\delta(x) = \int_{\xi-\delta}^{\xi+\delta} e^{-i\eta x}\big(\hat{K}(\xi) - \hat{K}(\eta)\big)\,d\eta.$$

The assumption (9) means that $\hat{K}'(\eta)$ exists and $|\hat{K}'(\eta)| \leqq M$ for all ξ, and with (13) this gives

$$(14) \qquad |E_\delta(x)| \leqq M\delta^2.$$

We need a slightly better estimate than this for large x which we obtain from (13) by integration by parts, obtaining

$$(15) \qquad |E_\delta(x)| \leqq 4M\delta x^{-1}.$$

Hence

$$(16) \qquad \int_{-\infty}^{\infty} |E_\delta(x)|^p\,dx \leqq \int_{|x|<4/\delta} (M\delta^2)^p\,dx + \int_{|x|>4/\delta} (4M\delta x^{-1})^p\,dx = 0(\delta^{2p-1}).$$

Define $u_\delta = v_\delta/\|v_\delta\|$, and use (11) and (16) to complete the proof.

Theorem 1. *Let $K \in L^1(\mathbf{R})$. Let $1<p<\infty$. Then the spectrum of $K*$ as an operator on $L^p(\mathbf{R})$ is the closure of the range of $\hat{K}$ on $\mathbf{R}$.*

Proof. Denote the closure of the range of $\hat{K}$ by σ. By Corollary 1 the spectrum is contained in σ. It suffices then to show that if $\lambda=\hat{K}(\xi)$ for some $\xi\in\mathbf{R}$ then λ is in the spectrum of $K*$. If $p=\infty$, the function $e^{-i\xi x}$ is an eigenvector of $K*$ with eigenvalue $\hat{K}(\xi)$, proving that $\sigma(K*;L^\infty)=\sigma$. For $p=1$, the result follows by taking adjoints reducing to $p=\infty$. Finally if $1<p<\infty$, we use Lemma 2 as follows: let $K_n(x)=K(x)$ if $|x|\leqq n$ and zero otherwise. Then K_n satisfies (9) so there is a $u_n\in L^p$ with $|u_n|=1$ and

$$\|K_n*u_n-\hat{K}_n(\xi)\,u_n\|_p<1/n. \tag{17}$$

We also have

$$\|K_n*-K*\|_p\leqq\|K_n-K\|_1=\int\limits_{|x|\geqq n}|K(x)|\,dx=\varepsilon_n, \tag{18}$$

and

$$|\hat{K}_n(\xi)-\hat{K}(\xi)|\leqq\varepsilon_n, \tag{19}$$

where $\varepsilon_n\to 0$ as $n\to\infty$.

Combining (17), (18), (19), we find that

$$\|K*u_n-\hat{K}(\xi)\,u_n\|_p\to 0\quad\text{as}\quad n\to\infty \tag{20}$$

which shows that $\hat{K}(\xi)\in\sigma(K*;L^p(\mathbf{R}))$.

Corollary 2. Let k be a measurable function on $(0,\infty)$ satisfying

$$\int\limits_0^\infty|k(s)|\,s^{-1/p}\,ds<\infty. \tag{21}$$

Let T be defined as in (1) and $\hat{k}$ as in (6). Then the spectrum of T as an operator on $L^p(0,\infty)$ is the closure of the range of $\hat{k}\left(-\frac{1}{p}+i\xi\right)$ as ξ varies over $\mathbf{R}$.

Proof. Let $K(x)=k(e^{-x})e^{-x/q}$ where $q=p/(p-1)$. Then $K\in L^1(\mathbf{R})$. For any $f\in L^p(0,\infty)$, let $Qf(x)=f(e^x)e^{x/p}$. Then Q is an isometry of $L^p(0,\infty)$ onto $L^p(\mathbf{R})$. Furthermore $QTQ^{-1}u(x)=K*u(x)$ for all $u\in L^p(\mathbf{R})$. Thus $\sigma(T;L^p(0,\infty))=$ $=\sigma(K*;L^p(\mathbf{R}))$ which is the closure of the range of $\hat{K}$ by Theorem 1. However, $\hat{K}(\xi)=\hat{k}\left(-\frac{1}{p}+i\xi\right)$.

Remarks 1. The proof of Theorem 1 for $p=1$ could be accomplished by noting that the norm of the operator $K*$ on $L^1(\mathbf{R})$ is the same as the L^1 norm of the function K, so the algebra of operators $K*$ is isometric with $L^1(\mathbf{R})$. The proof used for $1<p<\infty$ could also be modified to treat the case $p=1$.

2. Note that we made use of the Fourier transform only for $K\in L^1(\mathbf{R})$ and not for elements of $L^p(\mathbf{R})$. The usual proof of Theorem 1 for $p=2$ uses the fact

that the Fourier transform is a unitary operator on $L^2(\mathbf{R})$ so that $K*$ is unitarily equivalent to multiplication by $\hat{K}(\zeta)$. Such a proof is not available for $p \neq 2$.

3. Our Corollary 2 contains the result of Leibowitz quoted in the introduction.

3. Point spectrum. Suppose that K satisfies the conditions of Theorem 1. The next theorem determines conditions under which a value λ will be in the point spectrum of $K*$ acting on $L^p(\mathbf{R})$. We denote the point spectrum by $\pi(K*; L^p)$. Our conditions are necessary and sufficient only in case $p=1, 2$ or ∞. In contrast to Theorem 1, we need the Fourier transform of elements of $L^p(\mathbf{R})$. We recall that if $1 \leqq p \leqq 2$, and $u \in L^p$, then $\hat{u} \in L^q$, while if $2 < p \leqq \infty$, $\hat{u}$ is a tempered distribution [7, p. 142 and p. 146]. The results of Theorem 2 can be translated into results for operators T of the form (1).

Theorem 2. *Let $K \in L^1(\mathbf{R})$ and for each complex number λ, let $E_\lambda = \{\xi : \hat{K}(\xi) = \lambda\}$. Then*

(a) *$\lambda \in \pi(K*; L^1)$ if and only if E_λ contains an interval,*

(b) *if $1 < p < 2$ and if E_λ is of measure zero, then $\lambda \notin \pi(K*; L^p)$, while if E_λ contains an interval then $\lambda \in \pi(K*; L^p)$,*

(c) *$\lambda \in \pi(K*; L^2)$ if and only if E_λ has positive measure,*

(d) *if $2 < p < \infty$ and if E_λ is a finite set then $\lambda \notin \pi(K*; L^p)$, while if E_λ is of positive measure then $\lambda \in \pi(K*; L^p)$,*

(e) *$\lambda \in \pi(K*; L^\infty)$ if and only if E_λ is non-empty.*

Proof. Suppose that E_λ contains an interval $(a-\delta, a+\delta)$. Let $F(\xi) = \max(1-|\xi|, 0)$ and $u(x) = \int_R F((\xi-a)/\delta) e^{-i\xi x} d\xi$. Then $u \in L^p(\mathbf{R})$ for all $p \geqq 1$, and since $F((\xi-a)/\delta) = 0$ for $|\xi - a| > \delta$, we readily check that $K*u = \lambda u$. Similarly, if E_λ is of positive measure so contains a subset E of finite positive measure, then let $u(x) = \int_E e^{-i\xi x} d\xi$. Since χ_E is in L^q for $1 \leqq q \leqq 2$, we have $u \in L^p$ for $2 \leqq p \leqq \infty$, and as above, $K*u = \lambda u$. These remarks prove one direction of each of (a) to (d).

Conversely, suppose that u is in L^p and $K*u = \lambda u$. If $p=1$, this implies that

$$\hat{K}(\xi)\hat{u}(\xi) = \lambda \hat{u}(\xi) \tag{22}$$

for all ξ, and since $\hat{u}$ is continuous and vanishes except on E_λ by (22), it will vanish identically unless E_λ contains an interval. This proves (a), since $\hat{u}(\xi) = 0$ for all ξ implies that $u = 0$ a.e.

If $1 < p \leqq 2$, equation (22) is valid a.e. so that $\hat{u}(\xi)$ vanishes for almost all $\xi \notin E_\lambda$, and hence vanishes a.e. if E_λ is of measure zero. By the uniqueness theorem $\hat{u}(\xi) = 0$ a.e. implies that $u = 0$ a.e. This completes the proof of (b) and (c).

If $2 < p \leqq \infty$ and $K*u = \lambda u$ for $u \in L^p$, then (22) holds as a statement about tempered distributions. If φ is a testing function with support contained in an interval

I in the complement of E_λ, then there is a testing function ψ such that $(\hat{K}-\lambda)\psi = \varphi$. To see this, note that there is a $v\in L^1$ such that $\hat{v}(\xi) = (\hat{K}(\xi)-\lambda)^{-1}$ for $\xi\in I$ [5, p. 29]. Now let $\hat{\psi}=v*\hat{\varphi}$, and invert the Fourier transform to obtain ψ. Using

$$(\hat{K}-\lambda)\,\hat{u} = 0,$$

we have

$$0 = \langle(\hat{K}-\lambda)\,\hat{u},\psi\rangle = \langle\hat{u},(\hat{K}-\lambda)\,\psi\rangle = \langle\hat{u},\varphi\rangle.$$

This shows that the support of $\hat{u}$ is contained in E_λ. If E_λ is finite then Theorem 4. 12 and Theorem 4. 11 of [7, p. 152] show that $\hat{u}$ is a finite linear combination of point measures. But then $u\notin L^p$ if $p<\infty$. This contradiction shows that $\lambda\notin\pi(K*;L^p)$ and completes the proof of (d).

The proof of (e) is left to the reader.

4. Norms. According to Corollary 1, the spectral radius of a T given by (1) as an operator on $L^p(0,\infty)$ is given by

$$r_p(T) = \max_{\infty<\xi<\infty}\left|\hat{k}\left(-\frac{1}{p}+i\xi\right)\right|.$$

This is also the norm of T in case $p=2$, since T is a normal operator. This can also be proved directly using the Fourier transform as in Kober [8]. For $p=1$ or ∞, the norm of T is given by $N_p(k)$ of (2). If we associate with T the convolution operator $K*$ as in Corollary 2, then $N_p(k)=\|K\|_1=\|K*\|_1=\|K*\|_\infty$, and $r_p(T)=\max|\hat{K}(\xi)|= =\|K*\|_2$. Thus the Riesz—Thorin convexity theorem shows that in general

$$\|T\|_p \leqq N_p(k)^\gamma r_p(T)^{1-\gamma} \quad \text{where} \quad \gamma = |2-p|/p. \tag{23}$$

5. An example. Let T defined by the following expression:

$$Tf(t) = \int_0^1 s^{-1+(2/p)}f(ts)\,ds - \int_1^\infty s^{-1}f(ts)\,ds.$$

Then $\hat{k}\left(-\frac{1}{p}+i\xi\right) = -2i\xi/(p^{-2}+\xi^2)$, and hence $r_p(T)=p$, and the spectrum of T on $L^p(0,\infty)$ is the set $\{i\eta:|\eta|\leqq p\}$. According to Theorem 2, there is no point spectrum if $p<\infty$, while if $p=\infty$, the whole spectrum consists of point spectrum. We do not know the value of $\|T\|_p$ but it is easy to compute $N_p(k)=2p$, and hence (23) gives the estimate

$$\|T\|_p \leqq 2^{|p-2|/p}p$$

and obviously $\|T\|_p\geqq r_p(T)=p$. It would be interesting to show that $\|T\|_p>p$ if $p\neq 2$.

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Общие теоремы о факторизации оператор-функций относительно контура
II. Обобщения

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Эта статья является продолжением статьи авторов [1]. В ней получены различные обобщения теоремы из [1] о факторизации голоморфных оператор-функций относительно контура. В этой статье, в частности, доказываются теоремы, сформулированные во введении к [1]. Здесь используются определения и обозначения из [1].

Статья состоит из семи параграфов. Четвертый параграф *) носит вспомогательный характер, в нем вводятся специальные классы алгебр оператор-функций и банаховых пространств вектор-функций. В пятом параграфе доказывается основная теорема о факторизации оператор-функций в банаховых алгебрах. В шестом параграфе приводится ряд дополнений к основной теореме, а в седьмом излагаются обобщения этой теоремы. Отдельный восьмой параграф посвящен факторизации относительно контура оператор-функций, удовлетворяющих условию Гельдера. В девятом параграфе устанавливается одна теорема об устойчивости при малых возмущениях суммарного индекса. В небольшом десятом параграфе приводится одна теорема о неполной факторизации.

§ 4. Распадающиеся алгебры и связанные с ними пространства вектор-функций

В этом параграфе и в дальнейшем используются основные обозначения, введенные в § 1.

1. Обозначим через $\mathfrak{C}=\mathfrak{C}(\Gamma, L(\mathfrak{B}))$ произвольную банахову алгебру непрерывных оператор-функций $A\colon \Gamma \to L(\mathfrak{B})$, обладающую следующим свойством:

а) *для любой оператор-функций* $A \in \mathfrak{C}$ *имеет место соотношение*

$$\max_{\zeta \in \Gamma} \|A(\zeta)\|_{\mathfrak{B}} \leqq C\|A\|_{\mathfrak{C}},$$

где C *— некоторая константа, не зависящая от* A.

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*) Он являетсяп ервым в этой части статьи. Во всей статье нумерация параграфов сквозная.

Обозначим через $\mathfrak{C}^+ = \mathfrak{C}^+(\Gamma, L(\mathfrak{B}))$ множество всех оператор-функций из $\mathfrak{C}$, допускающих продолжения, голоморфные в F^+ и непрерывные вплоть до контура Γ. Очевидно, $\mathfrak{C}^+$ является замкнутой подалгеброй алгебры $\mathfrak{C}$. Аналогично через $\mathfrak{C}^- = \mathfrak{C}^-(\Gamma, L(\mathfrak{B}))$ обозначается подалгебра всех оператор-функций из $\mathfrak{C}$, допускающих продолжения, голоморфные в F^- и непрерывные на $\overline{F^-}$. Условимся еще в следующем обозначении:

$$\mathfrak{C}_0^- = \{A : A \in \mathfrak{C}^-, A(\infty) = 0\}.$$

Очевидно, пересечение $\mathfrak{C}^+ \cap \mathfrak{C}_0^-$ состоит только из нуля.

Алгебра $\mathfrak{C}$ называется *распадающейся*, если $\mathfrak{C}$ является прямой суммой подалгебр $\mathfrak{C}_0^-$ и $\mathfrak{C}^+ : \mathfrak{C} = \mathfrak{C}_0^- \dotplus \mathfrak{C}^+$.

Факторизация относительно контура Γ оператор-функции $A \in \mathfrak{C}: A = A_- D A_+$ называется *факторизацией в* $\mathfrak{C}$, если $A_-^{\pm 1} \in \mathfrak{C}^-$ и $A_+^{\pm 1} \in \mathfrak{C}^+$.

Из упоминавшейся в § 1 теоремы о факторизации элементов, близких к единичному, в абстрактных банаховых алгебрах ([2], гл. 1, лемма 5. 1) вытекает следующая лемма.

Лемма 4. 1. *Пусть алгебра* $\mathfrak{C}$ *распадается и имеет единицу* I. *Тогда существует константа* $\delta > 0$,[1]) *такая, что любая оператор-функция* $A \in \mathfrak{C}$, *удовлетворяющая условию* $\|A - I\|_{\mathfrak{C}} < \delta$, *допускает каноническую факторизацию в* $\mathfrak{C}$.

В дальнейшем будем рассматривать алгебры $\mathfrak{C}$, которые обладают еще следующими дополнительными свойствами:

б) *если все значения* $A(\zeta)$ $(\zeta \in \Gamma)$ *оператор-функции* $A \in \mathfrak{C}$ *принадлежат* $GL(\mathfrak{B})$, *то оператор-функция* $A^{-1} \in \mathfrak{C}$;

в) *всякая голоморфная функция* $A : \Gamma \to L(\mathfrak{B})$ *принадлежит алгебре* $\mathfrak{C}$. *Множество всех таких функций плотно в* $\mathfrak{C}$.

Приведем пример распадающейся алгебры. Пусть Γ_0 — единичная окружность и $\mathfrak{W} = \mathfrak{W}(L(\mathfrak{B}))$ — винеровская алгебра оператор-функций $A(\zeta)$ $(\zeta \in \Gamma_0)$, разлагающихся в абсолютно сходящийся степенной ряд

$$A(\zeta) = \sum_{j=-\infty}^{\infty} \zeta^j A_j \qquad (A_j \in L(\mathfrak{B}))$$

с нормой

$$\|A\|_{\mathfrak{W}} = \sum_{j=-\infty}^{\infty} \|A_j\|_{\mathfrak{B}}.$$

Очевидно, алгебра $\mathfrak{W}$ обладает свойствами а) и в). Проверка свойства б) проводится с помощью следующей леммы.

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[1]) Отметим, что $\delta = \min\{\|P\|^{-1}, \|Q\|^{-1}\}$, где P — проектор, проектирующий пространство $\mathfrak{C}$ на $\mathfrak{C}^+$ параллельно $\mathfrak{C}^-$, $Q = I - P$.

Лемма 4. 2. *Пусть* $\mathfrak{E}$ — *банахова алгебра непрерывных оператор-функций* $A\colon \Gamma \to L(\mathfrak{B})$, *которая обладает свойством* а), *и пусть* $Z_{\mathfrak{E}}$ — *множество всех непрерывных скалярных функций* φ *на* Γ, *для которых функции* $\varphi(\zeta) I\,(\zeta \in \Gamma)$ *принадлежат алгебре* $\mathfrak{E}$. *Если алгебра* $Z_{\mathfrak{E}}$ *содержит все рациональные функции с полюсами вне* Γ *и множество функций вида*

$$\sum_{j=1}^{n} \varphi_j(\zeta) X_j \qquad (X_j \in L(\mathfrak{B}),\ \varphi_j \in Z_{\mathfrak{E}})$$

плотно в $\mathfrak{E}$, *то алгебра* $\mathfrak{E}$ *обладает свойством* б).

Лемма 4. 2 легко выводится из одной общей теоремы Бохнера—Филлиса—Аллана [3, 4]. Это доказательство приведено в статье [5] и поэтому здесь опускается.

В дальнейшем понадобится следующая лемма о распадающихся алгебрах.

Лемма 4. 3. *Пусть* $\mathfrak{E}$ — *распадающаяся алгебра, обладающая свойствами* а), б) *и* в).

Для любой оператор-функции $A\colon \Gamma \to GL(\mathfrak{B})$, *принадлежащей алгебре* $\mathfrak{E}$, *существуют оператор-функции* $E_{\pm}\colon \bar{F}^{\pm} \to GL(\mathfrak{B})$ *такие, что* $E_+, E_+^{-1} \in \mathfrak{E}^+$, $E_-, E_-^{-1} \in \mathfrak{E}^-$ *и оператор-функция* $E_- A E_+$ *голоморфна на* Γ.

Доказательство этой леммы проводится так же, как доказательство аналогичной леммы 2. 3. При этомс сылки на лемму 1. 2 необходимо заменить с сылками на свойство в) алгебры $\mathfrak{E}$, а ссылки на лемму 1. 1 ссылкой на лемму 4. 1.

2. Пусть G — некоторое окрытое подмножество расширенной комплексной плоскости, граница ∂G которого состоит из конечного числа замкнутых спрямляемых жордановых кривых. Следуя книге Г. М. Голузина [6] (см. гл. X, § 5), через $E_1(G)$ обозначим класс скалярных голоморфных на G функций $\varphi(\zeta)\ (\zeta \in G)$ (обращающихся в нуль на бесконечности, если G неограничено), для которых существует последовательность контуров $\gamma_n \subseteq G$, сходящаяся к ∂G, такая, что

$$\sup_n \int_{\gamma_n} |\varphi(\zeta)|\,|d\zeta| < \infty.$$

Как известно (см. [6], гл. X, § 5), каждая функция $\varphi(z)\ (z \in G)$ из $E_1(G)$ почти всюду на ∂G имеет предельные значения $\varphi(\zeta)$ вдоль некасательных путей, причем функция $\varphi(\zeta)\ (\zeta \in \partial G)$ обладает следующими свойствами: $\varphi(\zeta) \in L_1(\partial G)$,

$$\varphi(z) = \frac{1}{2\pi i} \int_{\partial G} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta \quad (z \in G), \qquad \int_{\partial G} \varphi(\zeta)\, d\zeta = 0.$$

Обозначим через $\mathscr{L}_1=\mathscr{L}_1(\Gamma, \mathfrak{B})$ банахово пространство сильно измеримых вектор-функций $f\colon \Gamma\to\mathfrak{B}$, для которых

$$\|f\|_{\mathscr{L}_1} = \int_\Gamma \|f(\zeta)\|_{\mathfrak{B}}|d\zeta| < \infty.$$

Будем говорить, что вектор-функция $f\in\mathscr{L}_1$ принадлежит классу $\mathscr{E}_1^\pm=\mathscr{E}_1^\pm(\Gamma, \mathfrak{B})$, если для всех функционалов $a\in\mathfrak{B}^*$ скалярная функция $\langle f(\zeta), a\rangle$ принадлежит классу $E_1(F^\pm)$.

Пусть A — непрерывная на Γ оператор-функция со значениями из $L(\mathfrak{B})$. Через $\mathscr{K}$ обозначим оператор умножения на A в пространстве $\mathscr{L}_1$:

$$(\mathscr{A}f)(\zeta) = A(\zeta)f(\zeta) \qquad (\zeta\in\Gamma, f\in\mathscr{L}_1).$$

3. В дальнейшем будет предполагаться, что $\mathscr{B}$ — некоторое линейное подмножество пространства $\mathscr{L}_1$, которое со своей нормой является банаховым пространством.

Если $\mathfrak{C}=\mathfrak{C}(\Gamma, L(\mathfrak{B}))$ — некоторая банахова алгебра непрерывных оператор-функций $A\colon\Gamma\to L(\mathfrak{B})$, то пространство $\mathscr{B}$ называется $\mathfrak{C}$ — пространством, если оно обладает следующими свойствами.

α) *Для любой оператор-функции* $A\in\mathfrak{C}$ *имеет место* $\mathscr{A}(\mathscr{B})\subseteq\mathscr{B}$ *и сужение оператора* $\mathscr{A}$ на $\mathscr{B}$ *является ограниченным оператором в* $\mathscr{B}$.

β) *Для любого Γ-кольца K множество* $C_\omega(K, \mathfrak{B})$ *принадлежит* $\mathscr{B}$, *причем для любой функции* $f\in C_\omega(K, \mathfrak{B})$,

$$\|f\|_{\mathscr{B}} \leqq r_K \|f\|_{C_\omega(K,\mathfrak{B})},$$

где r_K — *константа, зависящая только от* K.[2])

Пространство $\mathscr{B}$ назовем *распадающимся*, если линейные множества $\mathscr{B}^+ \stackrel{\mathrm{def}}{=} \mathscr{B}\cap\mathscr{E}_1^+$ и $\mathscr{B}^- \stackrel{\mathrm{def}}{=} \mathscr{B}\cap\mathscr{E}_1^-$ являются замкнутыми подпространствами $\mathscr{B}$ и $\mathscr{B} = \mathscr{B}^-\dotplus\mathscr{B}^+$. Если пространство $\mathscr{B}$ распадается, то обозначим через $\mathscr{P}$ проектор, проектирующий пространство $\mathscr{B}$ на $\mathscr{B}^+$ параллельно $\mathscr{B}^-$, а через $\mathscr{Q}$ обозначим дополнительный проектор $\mathscr{Q} = \mathscr{I}-\mathscr{P}$.

4. Пусть $\mathfrak{B}=\mathfrak{H}$ — гильбертово пространство, Γ — контур типа Ляпунова $\mathscr{L}_2(\Gamma, \mathfrak{H})$ — гильбертово пространство сильно измеримых вектор-функций $f\colon\Gamma\to\mathfrak{H}$, для которых

$$\|f\|^2_{\mathscr{L}_2} \stackrel{\mathrm{def}}{=} \int_\Gamma \|f(\zeta)\|^2_{\mathfrak{H}}|d\zeta| < \infty.$$

Очевидно, $\mathscr{L}_2(\Gamma, \mathfrak{H})$ является $\mathfrak{C}$ — пространством для любой алгебры $\mathfrak{C}=\mathfrak{C}(\Gamma, L(\mathfrak{H}))$ непрерывных оператор-функций $A\colon\Gamma\to L(\mathfrak{H})$. Кроме того, нетрудно

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[2]) Кроме того, предполагается, что норма в пространстве $\mathscr{B}$ сильнее чем в пространстве $\mathscr{L}_1$.

показать (подробно это сделано в [5], гл. Ш, лемма 1. 1), что пространство $\mathscr{L}_2(\Gamma, \mathfrak{H})$ распадается. Отметим также равенства

$$\|\mathscr{P}\|_{\mathscr{L}_2} = \|P\|_{L_2} \quad \text{и} \quad \|\mathscr{Q}\|_{\mathscr{L}_2} = \|\mathscr{Q}\|_{L_2(\Gamma)}, \tag{4. 1}$$

где $L_2(\Gamma)$ — пространство L_2 скалярных функций на Γ, P — проектор, проектирующий $L_2(\Gamma)$ на $L_2^+(\Gamma) \overset{\text{def}}{=} L_2(\Gamma) \cap E_1(F^+)$ параллельно $L_2^-(\Gamma) \overset{\text{def}}{=} L_2(\Gamma) \cap E_1(F^-)$[3]) а $Q = I-P$. Из (4. 1) вытекает, в частности, что в случае, когда Γ является окружностью, имеет место равенство

$$\|\mathscr{P}\|_{\mathscr{L}_2} = \|\mathscr{Q}\|_{\mathscr{L}_2} = 1. \tag{4. 2}$$

5. Приведем второй пример. Пусть $\mathfrak{B}$ — банахово пространство и Γ — контур Ляпунова. Обозначим через $\mathscr{L}_p = \mathscr{L}_p(\Gamma, \mathfrak{B})$ $(1 < p < \infty)$ банахово пространство измеримых вектор-функций $f: \Gamma \to \mathfrak{B}$, для которых

$$\|f\|_{\mathscr{L}_p}^p = \int_\Gamma \|f(\zeta)\|^p |d\zeta| < \infty.$$

Очевидно, $\mathscr{L}_p(\Gamma, \mathfrak{B})$ является $\mathfrak{C}$-пространством для любой алгебры $\mathfrak{C} = \mathfrak{C}(\Gamma, L(\mathfrak{B}))$ непрерывных оператор-функций $A: \Gamma \to L(\mathfrak{B})$. Однако пространство $\mathscr{L}_p(\Gamma, \mathfrak{B})$ не всегда распадается (см. пример в [5], гл. Ш, § 2), но в случае, когда пространство $\mathfrak{B}$ само является пространством типа L_p, пространство $\mathscr{L}_p(\Gamma, \mathfrak{B})$ $(1 < p < \infty)$ распадается. Поясним это подробнее.

Пусть $[S, \mathfrak{S}, \mu]$ — некоторое пространство с мерой и $\mathfrak{L}_p = \mathfrak{L}_p(S)$ $(1 < p < \infty)$ — банахово пространство скалярных измеримых функций на S с нормой

$$\|\varphi\|_{\mathscr{L}_p}^p = \int_S |\varphi(s)|^p \mu(ds).$$

Как показано в [5] (лемма 2. 1, гл. Ш), пространство $\mathscr{L}_p(\Gamma, \mathfrak{L}_p)$ распадается для любого $1 < p < \infty$, причем имеют место равенства

$$\|\mathscr{P}\|_{\mathscr{L}_p(\Gamma, \mathfrak{L}_p)} = \|P\|_{L_p(\Gamma)} \quad \text{и} \quad \|\mathscr{Q}\|_{\mathscr{L}_p(\Gamma, \mathfrak{L}_p)} = \|Q\|_{L_p(\Gamma)}, \tag{4. 3}$$

где $L_p(\Gamma)$ — пространство L_p скалярных функций на Γ, P — проектор, проектирующий $L_p(\Gamma)$ на $L_p^+(\Gamma) \overset{\text{def}}{=} L_p(\Gamma) \cap E_1(F^+)$ параллельно $L_p^-(\Gamma) = L_p(\Gamma) \cap E_1(F^-)$,[4]) а $Q = I-P$.

6. Перейдем к третьему примеру. Пусть $\mathfrak{B}$ — произвольное банахово пространство и Γ-гладкий контур. Обозначим через $H_\alpha(\Gamma, \mathfrak{B})$ банахово пространство

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[3]) Как известно, пространство $L_2(\Gamma)$ распадается в прямую сумму подпространств $L_2^+(\Gamma)$ и $L_2^-(\Gamma)$, которые в случае окружности являются ортогональными.

[4]) Как известно, пространство $L_p(\Gamma)$ распадается в прямую сумму его подпространств $L_p^+(\Gamma)$ и $L_p^-(\Gamma)$ при $1 < p < \infty$.

функций $f: \Gamma \to \mathfrak{B}$, удовлетворяющих условию Гельдера с показателем α $(0<\alpha<1)$, с нормой

$$\|f\|_\alpha = \max_{\zeta \in \Gamma} \|f(\zeta)\| + \sup_{\zeta_1, \zeta_2 \in \Gamma; \zeta_1 \neq \zeta_2} \frac{\|f(\zeta_1)-f(\zeta_2)\|}{|\zeta_1-\zeta_2|^\alpha}.$$

Как отмечено в [7], пространство $H_\alpha(\Gamma, \mathfrak{B})$ распадается. Отметим, что $H_\alpha(\Gamma, L(\mathfrak{B}))$ является банаховой алгеброй.

Пространство $H_\alpha(\Gamma, \mathfrak{B})$ является $H_\alpha(\Gamma, L(\mathfrak{B}))$-пространством. В самом деле, условие α), очевидно, выполняется. Пусть теперь для некоторого Γ-кольца K последовательность $f_n \in C_\omega(K, \mathfrak{B})$ сходится по норме $C_\omega(K, \mathfrak{B})$ к f. Отсюда следует, в частности, равномерная сходимость на Γ последовательностей f_n и $df_n/d\zeta$ к f и $df/d\zeta$ соответственно. Последнее влечет за собой сходимость последовательности f_n к f по норме $H_\alpha(\Gamma, \mathfrak{B})$. Следовательно, условие β') также выполняется.

7. Рассмотрим четвертый пример. Пусть $\mathfrak{B}$-банахово пространство и $\mathfrak{C}$-распадающаяся алгебра непрерывных оператор-функций $A: \Gamma \to L(\mathfrak{B})$ со свойствами а), б) и в). Предполагается, что, кроме того, для каждого Γ-кольца K норма пространства $C_\omega(K, \mathfrak{B})$ сильнее нормы $\mathfrak{C}$.

Ниже будет показано, что для всякой такой алгебры существует распадающееся $\mathfrak{C}$-пространство $\mathscr{B}$, которое строится естественным образом.

Обозначим через $\mathscr{B}(\mathfrak{C})$ линейное множество всех вектор-функций $f: \Gamma \to \mathfrak{B}$ вида $f(\zeta)=A(\zeta)x$ (x — фиксированный вектор из $\mathfrak{B}$ а A — произвольная оператор-финкция из $\mathfrak{C}$). Сопоставим каждой вектор-функции $f \in \mathscr{B}(\mathfrak{C})$ оператор-функцию $T_f: \Gamma \to L(\mathfrak{B})$, определенную равенством

$$T_f(\zeta)x = \langle x, a \rangle f(\zeta) \qquad (\zeta \in \Gamma, x \in \mathfrak{B}),$$

где a — фиксированный функционал из $\mathfrak{B}^*$, для которого $\langle x, a \rangle = 1$.

Очевидно, оператор-функция T_f принадлежит алгебре $\mathfrak{C}$ и имеет вид $T_f = A(\zeta)R$, где $R = \langle \cdot, a \rangle x$ и $A \in \mathfrak{C}$. Множество всех оператор-функций вида T_f, где f пробегает $\mathscr{B}(\mathfrak{C})$, образует замкнутое подпространство $\mathfrak{C}$. Следовательно, линейное множество $\mathscr{B}(\mathfrak{C})$ является банаховым пространством с нормой

$$\|f\|_{\mathscr{B}(\mathfrak{C})} \stackrel{\text{def}}{=} \|T_f\|_{\mathfrak{C}} \qquad (f \in \mathscr{B}(\mathfrak{C})).$$

Без труда проверяется, что пространство $\mathscr{B}$ распадается и обладает свойством α) (из определения $\mathfrak{C}$-пространства). Пусть теперь K-некоторое Γ-кольцо. Если f — любая функция из $C_\omega(K, \mathfrak{B})$ то в силу леммы 1. 3 существует голоморфная оператор-функция $A: \Gamma \to L(\mathfrak{B})$, такая, что $f(\zeta) = A(\zeta)x$. Так как согласно свойству в) алгебры $\mathfrak{C}$ оператор-функция A принадлежит $\mathfrak{C}$, то $f \in \mathscr{B}(\mathfrak{C})$. Следовательно, $C_\omega(K, \mathfrak{B}) \subseteq \mathscr{B}(\mathfrak{C})$. Из того что норма алгебры $C_\omega(K, \mathfrak{B})$ сильнее нормы $\mathfrak{C}$, вытекает, что норма $C_\omega(K, \mathfrak{B})$ сильнее нормы пространства $\mathscr{B}(\mathfrak{C})$. Таким образом, пространство $\mathscr{B}(\mathfrak{C})$ является $\mathfrak{C}$-пространством.

§ 5. Общая теорема

1. Одной из основых в статье является следующая теорема.

Теорема 5. 1. *Пусть* $\mathfrak{C}=\mathfrak{C}(\Gamma, L(\mathfrak{B}))$ — *распадающаяся банахова алгебра непрерывных оператор-функций, обладающая свойствами* а), б) *и* в), *и* $\mathscr{B}$ — *некоторое распадающееся* $\mathfrak{C}$-*пространство.*

Оператор-функция $A:\Gamma\to GL(\mathfrak{B})$ *из алгебры* $\mathfrak{C}$ *допускает факторизацию в* $\mathfrak{C}$ *в том и только том случае, когда оператор* $\mathscr{P}\mathscr{A}$ *является* Φ-*оператором в* $\mathscr{B}^+$.

Доказательству теоремы 5. 1 предпошлем следующую лемму.

Лемма 5. 1. *Пусть* K *является* Γ-*кольцом и оператор-функция* $Z:\overline{K}\to GL(\mathfrak{B})$ *голоморфна на* $\overline{K}$. *Если*

$$Z(\zeta)g_+(\zeta)=f(\zeta)+g_-(\zeta)\qquad(\zeta\in\Gamma),\tag{5.1}$$

где $g_\pm\in E_1^\pm$ *и* $f\in C_\omega(K,\mathfrak{B})$, *то* $g_\pm\in C_\omega^\pm(K,\mathfrak{B})$.

Доказательство. Достаточно показать, что $g_+\in C_\omega(K,\mathfrak{B})$. Без ограничения общности можно считать, что $\Gamma_1\overset{\text{def}}{=}\partial K\cap F^-$ является параллельным контуром к контуру Γ. Контур Γ_1 ориентируем так, чтобы при его обходе множество K осталось справа.

Рассмотрим функции

$$g_-(\zeta)\overset{\text{def}}{=}-\frac{1}{2\pi i}\int_\Gamma\frac{g_-(\eta)}{\eta-\zeta}\,d\eta\qquad(\zeta\in F^-)$$

и

$$\tilde g_+(\xi)\overset{\text{def}}{=}-\frac{1}{2\pi i}\int_{\Gamma_1}\frac{Z^{-1}(\zeta)\big(f(\zeta)+g_-(\zeta)\big)}{\zeta-\xi}\,d\zeta\qquad(\xi\in F^+\cup K).$$

Очевидно, $\tilde g_+\in C_\omega^+(K,\mathfrak{B})$. Осталось показать, что $\tilde g_+=g_+$.

Покажем сначала, что для всех $a\in\mathfrak{B}^*$ и $\zeta\in F^+$ имеет место равенство

$$\int_{\Gamma\cup\Gamma_1}\frac{\langle Z^{-1}(\zeta)g_-(\zeta),a\rangle}{\zeta-\xi}\,d\zeta=0.\tag{5.2}$$

В силу леммы 1. 2 существуют оператор-функции

$$Y_n(\zeta)=\sum_{j=1}^{k_n}r_{nj}(\zeta)Y_{nj}\qquad(n=1,2,\ldots),$$

где r_{nj} — рациональные функции с полюсами вне $\overline{K}$ и $Y_{nj}\in L(\mathfrak{B})$, такие, что

$$\max_{\zeta\in\overline{K}}\|Z^{-1}(\zeta)-Y_n(\zeta)\|_{\mathfrak{B}}\leqq\frac{1}{n}.\tag{5.3}$$

Так как $g_-\in\mathscr{E}_1^-$, то функции

$$\langle Y_n(\zeta)g_-(\zeta),a\rangle=\sum_{j=1}^{k_n}\frac{r_{nj}(\zeta)}{\zeta-\xi}\langle g_-(\zeta),Y_{nj}^*a\rangle\qquad(n=1,2,\ldots)$$

принадлежат классу $E_1(K \cap F^-)$. Следовательно (см. (4. 2)),

$$\int_{\Gamma \cup \Gamma_1} \frac{\langle Y_n(\zeta)\, g_-(\zeta), a\rangle}{\zeta - \xi}\, d\zeta = 0.$$

Отсюда в силу (5. 3) вытекает равенство (5. 2).

Из (5. 2) и (5. 1) следуют равенства

$$\langle \tilde{g}_+(\xi), a\rangle = \frac{1}{2\pi i}\int_{\Gamma} \frac{\langle Z^{-1}(\zeta)\,[f(\zeta)+g_-(\zeta)], a\rangle}{\zeta-\xi}\, d\zeta = \frac{1}{2\pi i}\int_{\Gamma} \frac{\langle g_+(\zeta), a\rangle}{\zeta - \xi}\, d\zeta$$

для всех $a \in \mathfrak{B}^*$ и $\xi \in F^+$. Учитывая равенство (4. 1), а также принадлежность функции $\langle \tilde{g}_+(\cdot) - g_+(\cdot), a\rangle$ пространству $E_1(F^+)$, получаем, что для любого функционала $a \in \mathfrak{B}^*$ существует множество $\Gamma_a \subseteq \Gamma$ меры нуль, такое, что

$$\langle \tilde{g}_+(\zeta) - g_+(\zeta), a\rangle = 0 \quad \text{для всех} \quad \zeta \in \Gamma \setminus \Gamma_a \tag{5. 4}$$

Так как функция $\tilde{g}_+ - g_+$ сильно измерима, то существует сепарабельное подпространство $\mathfrak{N}$ пространства $\mathfrak{B}$ такое, что $\tilde{g}_+(\zeta) - g_+(\zeta) \in \mathfrak{N}$ почти всюду на Γ. В силу теоремы Хана—Банаха легко найти последовательность функционалов $a_n \in \mathfrak{B}^*$ $(n=1, 2, \ldots)$, такую, что для всех $x \in \mathfrak{N}$ имеет место равенство

$$\|x\| = \sup_n |\langle x, a\rangle|.$$

Отсюда и из (5. 4) получаем, что почти всюду на множестве $\Gamma \setminus \bigcup_1^\infty \Gamma_{a_n}$, т. е. почти всюду на Γ, имеет место равенство $\tilde{g}_+(\zeta) - g_+(\zeta) = 0$.

Лемма доказана.

2. Доказательство теоремы 5. 1. Пусть оператор-функция $A: \Gamma \to GL(\mathfrak{B})$ принадлежит алгебре $\mathfrak{C}$ и оператор $\mathscr{P}\mathscr{A}$ является Φ-оператором в $\mathscr{B}^+$. Обозначим через $E_\pm$ оператор-функции из леммы 4. 3 и через K — кольцо, такое, что оператор-функция $E_- A E_+$ голоморфна на замкнутом кольце $\bar{K}$ и

$$E_-(\zeta) A(\zeta) E_+(\zeta) \in GL(\mathfrak{B})$$

для всех $\zeta \in \bar{K}$. Так как операторы $\mathscr{P}\mathscr{E}_\pm$ обратимы в $\mathscr{B}^+$ $\left((\mathscr{P}\mathscr{E}_\pm)^{-1} = \mathscr{P}\mathscr{E}_\pm^{-1}\right)$, то и оператор $\mathscr{P}\mathscr{E}_- \mathscr{A} \mathscr{E}_+ = (\mathscr{P}\mathscr{E}_-)(\mathscr{P}\mathscr{A})(\mathscr{P}\mathscr{E}_+)$ является Φ-оператором в $\mathscr{B}^+$. Положим $\tilde{A} = E_- A E_+$.

Оператор $\mathscr{P}\tilde{\mathscr{A}}$ можно рассматривать как оператор, действующий в $C_\omega^+(K, \mathfrak{B})$. Через $\mathscr{P}\tilde{\mathscr{A}}|\mathscr{B}^+$ будем обозначать оператор $\mathscr{P}\tilde{\mathscr{A}}$, рассматриваемый в $\mathscr{B}^+$, а через $\mathscr{P}\tilde{\mathscr{A}}|C_\omega^+$ — оператор $\mathscr{P}\tilde{\mathscr{A}}$, рассматриваемый в $C_\omega^+(K, \mathfrak{B})$.

Для любого $f_+ \in C_\omega^+(K, \mathfrak{B}) \cap \operatorname{Im} \mathscr{P}\tilde{\mathscr{A}}|\mathscr{B}^+$ уравнение

$$\mathscr{P}\tilde{\mathscr{A}} g_+ = f_+$$

имеет решение $g_+ \in \mathscr{B}^+$, которое в силу леммы 5. 1 принадлежит пространству $C_\omega^+(K, \mathfrak{B})$. Следовательно,

$$\operatorname{Im} \mathscr{P}\tilde{\mathscr{A}}\,|C_\omega^+ = C_\omega^+(K, \mathfrak{B}) \cap \operatorname{Im} \mathscr{P}\tilde{\mathscr{A}}\,|\mathscr{B}_+. \tag{5. 5}$$

Так как множество $\operatorname{Im} \mathscr{P}\tilde{\mathscr{A}}|\mathscr{B}^+$ замкнуто по норме $\mathscr{B}^+$, а в силу свойства β) норма пространства $C^+_\omega(K, \mathfrak{B})$ сильнее нормы $\mathscr{B}_+$, то из (5. 5) вытекает замкнутость множества $\operatorname{Im} \mathscr{P}\tilde{\mathscr{A}}|C^+_\omega$ по норме $C^+_\omega(K, \mathfrak{B})$. Кроме того, из (5. 5) вытекает оценка $\dim \operatorname{Coker} \mathscr{P}\tilde{\mathscr{A}}|C^+_\omega \leqq \dim \operatorname{Coker} \mathscr{P}\tilde{\mathscr{A}}|\mathscr{B}^+ < \infty$.

Очевидно, $\operatorname{Ker} \mathscr{P}\tilde{\mathscr{A}}|C^+_\omega \leqq \operatorname{Ker} \mathscr{P}\tilde{\mathscr{A}}|\mathscr{B}^+$ и, следовательно, $\dim \operatorname{Ker} \mathscr{P}\tilde{\mathscr{A}}|C^+_\omega < \infty$. Таким образом, оператор $\mathscr{P}\tilde{\mathscr{A}}$ является Φ-оператором в пространстве $C_\omega(K, \mathfrak{B})$.

Следовательно, в силу теоремы 2. 1 оператор-функция $\tilde{\mathscr{A}}$ допускает факторизацию: $\tilde{A} = \tilde{A}_- D \tilde{A}_+$. Полагая $A_- = E_-^{-1}\tilde{A}_-$ и $A_+ = \tilde{A}_+ E_+^{-1}$, получаем факторизацию оператор-функции A: $A = A_- D A_+$.

Теорема 5. 1 доказана.[5])

3. Приведем один пример применения теоремы 5. 1. Пусть $\mathfrak{B}$ — банахово пространство. Обозначим через $\mathscr{L}_1(\mathfrak{B})$ банахово пространство всех сильно измеримых вектор-функций $f(t)$ $(-\infty < t < \infty)$ со значениями из $\mathfrak{B}$, для которых

$$\|f\|_{\mathscr{L}_1} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \|f(t)\|_{\mathfrak{B}}\, dt < \infty.$$

Через $\mathfrak{W}(\mathfrak{B})$ обозначим банахово пространство всех вектор-функций вида

$$a(\lambda) = a_0 + \int_{-\infty}^{\infty} e^{i\lambda t} f(t)\, dt \qquad (-\infty < \lambda < \infty,\ a_0 \in \mathfrak{B}, f \in \mathscr{L}_1(\mathfrak{B})) \tag{5. 6}$$

с нормой

$$\|a\|_{\mathfrak{M}} = \|a_0\|_{\mathfrak{B}} + \|f\|_{\mathscr{L}_1}.$$

Через $\mathfrak{W}_+(\mathfrak{B})$ обозначим подалгебру $\mathfrak{W}(\mathfrak{B})$, состоящую из вектор-функций вида (5. 6), для которых $f(t) \equiv 0$ при $t < 0$. Аналогично через $\mathfrak{W}_-(\mathfrak{B})$ обозначим подалгебру $\mathfrak{W}(\mathfrak{B})$, состоящую из вектор-функций вида (5. 6), для которых $f(t) \equiv 0$ при $t > 0$. Обозначим через $\mathscr{P}$ проектор, проектирующий $\mathfrak{W}(\mathfrak{B})$ на $\mathfrak{W}_+(\mathfrak{B})$ параллельно $\mathfrak{W}_-(\mathfrak{B})$.

Теорема 5. 2. *Пусть оператор-функция $A: (-\infty, +\infty) \to GL(\mathfrak{B})$ принадлежит банаховой алгебре $\mathfrak{W}(L(\mathfrak{B}))$. Для того чтобы оператор-функция A допускала факторизацию в $\mathfrak{W}(L(\mathfrak{B}))$, т. е. была представима в виде*

$$A(\lambda) = A_-(\lambda) D(\lambda) A_+(\lambda),$$

в котором A_-, $A_-^{-1} \in \mathfrak{W}_-(L(\mathfrak{B}))$, A_+, $A_+^{-1} \in \mathfrak{W}_+(L(\mathfrak{B}))$ и

$$D(\lambda) = \sum_{j=1}^{n} \left(\frac{\lambda - i}{\lambda + i}\right)^{\varkappa_j} P_j + P_0, \tag{5. 7}$$

где $P_1, \ldots, P_n$ — дизъюнктные одномерные проекторы, $P_0 = I - P_1 - \cdots - P_n$, $\varkappa_1 \geqq \varkappa_2 \geqq \cdots \geqq \varkappa_n$ — целые числа, отличные от нуля, необходимо и достаточно, чтобы оператор $\mathscr{P}\mathscr{A}$ был Φ-оператором в пространстве $\mathfrak{W}_+(\mathfrak{B})$.

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[5]) Необходимость условия теоремы очевидна.

Доказательство. Рассмотрим банахово пространство $\tilde{\mathfrak{B}}(\mathfrak{B})$, состоящее из всех вектор-функций вида $a\left(i\dfrac{1+\zeta}{1-\zeta}\right)$ $(|\zeta|=1)$, где $a(\lambda)\in\mathfrak{B}(\mathfrak{B})$ с нормой

$$\left\|a\left(i\frac{1+\zeta}{1-\zeta}\right)\right\|_{\tilde{\mathfrak{B}}} = \|a(\lambda)\|_{\mathfrak{B}}.$$

С помощью леммы 4. 2 легко доказывается, что алгебра $\tilde{\mathfrak{B}}(L(\mathfrak{B}))$ обладает свойствами а)—в). Алгебра $\tilde{\mathfrak{B}}_{\pm}(L(\mathfrak{B}))$ состоит из всех оператор-функций вида $A\left(i\dfrac{1+\zeta}{1-\zeta}\right)$, где $A(\lambda)\in\mathfrak{B}_{\pm}(L(\mathfrak{B}))$. Отсюда следует, что алгебра $\tilde{\mathfrak{B}}(L(\mathfrak{B}))$ является распадающейся.

Нетрудно видеть, что пространство $\tilde{\mathfrak{B}}(\mathfrak{B})$ является $\tilde{\mathfrak{B}}(L(\mathfrak{B}))$-пространством.

Таким образом, к оператор-функциям из алгебры $\tilde{\mathfrak{B}}(L(\mathfrak{B}))$ применима теорема 5. 1. Заменяя в полученной согласно теореме 5. 1. факторизации ζ на $(\lambda-i)/(\lambda+i)$, получим факторизацию (5. 7).

Теорема доказана.

§ 6. Дополнение к общей теореме

В теореме 5. 1 предполагается, что все значения оператор-функции A обратимы. В случае пространства $\mathscr{L}_p$ это предположение можно опустить. Имеет место следующая теорема.

Теорема 6. 1. *Пусть $\mathfrak{B}$ — банахово пространство и при некотором p $(1<p<\infty)$ пространство $\mathscr{L}_p(\Gamma,\mathfrak{B})$ распадается. Если для непрерывной оператор-функции $A:\Gamma\to L(\mathfrak{B})$ оператор $\mathscr{P}\mathscr{A}$ является Φ-оператором в пространстве $\mathscr{L}_p^+(\Gamma,\mathfrak{B})$, то все значения $A(\zeta)$ $(\zeta\in\Gamma)$ обратимы.*

Доказательство. Покажем сначала, что все операторы $A(\zeta)$ $(\zeta\in\Gamma)$ являются операторами регулярного типа. Допустим, что для некоторого $\zeta_0\in\Gamma$ оператор $A(\zeta_0)$ не является оператором регулярного типа.

Если $V\subseteqq\Gamma$, то обозначим через $\mathscr{L}_p(V,\mathfrak{B})$ подпространство всех функций $h\in\mathscr{L}_p(\Gamma,\mathfrak{B})$ со свойством $h(\zeta)\equiv 0$ $(\zeta\in\Gamma\setminus V)$. Покажем теперь, что существует дуга $U\subseteqq\Gamma$, содержащая точку ζ_0, такая, что для любых $h\in\mathscr{L}_p(U,\mathfrak{B})$ и $g\in\operatorname{Ker}\mathscr{P}\mathscr{A}$ имеет место оценка

$$\|h-g\|_{\mathscr{L}_p}\geqq\frac{1}{2}\|h\|_{\mathscr{L}_p}. \tag{6. 1}$$

Пусть $g_1,\ldots,g_k$ — базис конечномерного подпространства $\operatorname{Ker}\mathscr{P}\mathscr{A}$ и $C<\infty$ — такая константа, что

$$\sum_{j=1}^{k}|\alpha_j|\leqq C\left\|\sum_{j=1}^{k}\alpha_j g_j\right\|_{\mathscr{L}_p}.$$

Выберем дугу U настолько малой, чтобы

$$\max_{j=1,\ldots,k} \|g_j\|_U \leqq \frac{1}{4C},$$

где

$$\|f\|_U \stackrel{\text{def}}{=} \Big(\int_U \|f(\zeta)\|^p |d\zeta|\Big)^{1/p} \qquad (f \in \mathscr{L}_p(\Gamma, \mathfrak{B})).$$

Тогда для любого вектора $g = \sum \alpha_j g_j \in \operatorname{Ker} \mathscr{P}\mathscr{A}$

$$\|g\|_U \leqq \sum |\alpha_j|\, \|g_j\|_U \leqq \frac{1}{4C} \sum |\alpha_j| \leqq \frac{1}{4} \|g\|_{\mathscr{L}_p}. \tag{6.2}$$

Соотношение (6.1) в случае $\|g\|_{\mathscr{L}_p} \geqq 2\|h\|_{\mathscr{L}_p}$ тривиально, а в случае $\|g\|_{\mathscr{L}_p} \leqq 2\|h\|_{\mathscr{L}_p}$ оно следует из (6.2):

$$\|h-g\|_{\mathscr{L}_p} \geqq \|h\|_{\mathscr{L}_p} - \|g\|_U \geqq \|h\|_{\mathscr{L}_p} - \frac{1}{4} \|g\|_{\mathscr{L}_p} \geqq \frac{1}{2} \|h\|_{\mathscr{L}_p}.$$

Оператор $A(\zeta_0)$ не является оператором регулярного типа, т. е. существует последовательность векторов $x_n \in \mathfrak{B}$ со следующими свойствами:

$$\|x_n\| = 1 \quad \text{и} \quad \|A(\zeta_0)\, x_n\| < \frac{1}{n} \qquad (n = 1, 2, \ldots) \tag{6.3}$$

Выберем дугу $U_n \subseteqq U$ так, чтобы

$$\|A(\zeta)\, x_n\| < \frac{1}{n} \qquad (\zeta \in U_n). \tag{6.4}$$

Пусть r_n — рациональные функции, такие, что

$$\max_{\zeta \in \Gamma \setminus U_n} |r_n(\zeta)| < \frac{1}{n} \tag{6.5}$$

и

$$\int_{U_n} |r_n(\zeta)^p|\, |d\zeta| = 1. \tag{6.6}$$

Обозначим через Γ_0 связную компоненту контура Γ, которая содержит точку ζ_0. Контур Γ_0 разбивает комплексную плоскость на две области: G_0^1 и G_0^2. Обозначим через G_0^+ ту область из G_0^1 и G_0^2, которая в окрестности контура Γ_0 совпадает с множеством F^+. Перемножая некоторые конформные отображения, отображающие область на единичный круг (подробно это изложено в статье [8]), найдем голоморфные в F^+ и непрерывные на $\overline{F}^+$ скалярные функции φ_n со следующими свойствами:

1) $|\varphi_n(\zeta)| \leqq 1 (\zeta \in \Gamma)$,
2) $|\varphi_n(\zeta)| \equiv 1$ $(\zeta \in \Gamma_0)$,
3) функции $\varphi_n(\zeta) r_n(\zeta)$ голоморфны на F^+.

Положим

$$f_n(\zeta) = \varphi_n(\zeta)\, r_n(\zeta)\, x_n.$$

Очевидно, $f_n \in \mathscr{L}_p^+(\Gamma, \mathfrak{B})$.

Обозначим через χ_n характеристическую функцию дугу U_n. Тогда в силу (6. 1) и (6. 5) для любой функции $g \in \operatorname{Ker} \mathscr{P}\mathscr{A}$ имеет место соотношение

$$\|f_n - g\|_{\mathscr{L}_p} \geqq \|\chi_n f_n - g\|_{\mathscr{L}_p} - \|(1-\chi_n) f_n\|_{\mathscr{L}_p} \geqq \frac{1}{2}\|\chi_n f_n\| - \frac{1}{n}|\Gamma|,$$

где $|\Gamma|$-длина контура Γ. Так как в силу (6. 6)

$$\|x_n f_n\|_{\mathscr{L}_p} = \Big(\int_{U_n} \|\varphi_n(\zeta) r(\zeta) x_n\|^p |d\zeta|\Big)^{1/p} = 1,$$

то отсюда вытекает

$$\|f_n - g\|_{\mathscr{L}_p} \geqq \frac{1}{2} - \frac{|\Gamma|}{n} \qquad (n = 1, 2, \ldots; g \in \operatorname{Ker} \mathscr{P}\mathscr{A}). \tag{6. 7}$$

Кроме того,

$$\|\mathscr{P}\mathscr{A} f_n\|_{\mathscr{L}_p}^p \leqq \|\mathscr{P}\| \Big[\int_{\Gamma \setminus U_n} |r_n(\zeta)|^p \|A(\zeta) x_n\|^p |d\zeta| + \int_{U_n} |r_n(\zeta)|^p \|A(\zeta) x_n\|^p |d\zeta|\Big].$$

Следовательно, в силу (6. 4) и (6. 5)

$$\|\mathscr{P}\mathscr{A} f_n\|_{\mathscr{L}_p}^p \leqq \|\mathscr{P}\| \Big[\Big(\frac{1}{n}\Big)^p \int_\Gamma \|A(\zeta)\|^p |d\zeta| + \Big(\frac{1}{n}\Big)^p\Big] \xrightarrow[n\to\infty]{} 0. \tag{6. 8}$$

Соотношения (6. 7) и (6. 8) противоречат нормальной разрешимости оператора $\mathscr{P}\mathscr{A}$.

Допустим теперь, что для некоторого $\zeta_0 \in \Gamma$ оператор $A(\zeta_0)$ не обратим. Так как он нормального типа, то тогда существует вектор $x \neq 0$, такой, что $x \notin \operatorname{Im} A(\zeta_0)$, и функционал $a \in \mathfrak{B}^*$, такой, что $\langle x, a\rangle = 1$ и $\langle \operatorname{Im} A(\zeta_0), a\rangle = 0$.

Легко видеть, что существуют дуги $U_n \leqq \Gamma$ и непрерывные оператор-функции $A_n : \Gamma \to L(\mathfrak{B})$, для которых $\|A_n(\zeta) - A(\zeta)\| < \frac{1}{n}$ $(\zeta \in \Gamma)$ и $A_n(\zeta) = A(\zeta_0)$ $(\zeta \in U_n)$. Очевидно,

$$\lim_{n\to\infty} \|\mathscr{P}\mathscr{A}_n - \mathscr{P}\mathscr{A}\|_{\mathscr{L}_p} = 0. \tag{6. 9}$$

Функции вида $\varphi_+(\zeta)x$, где $\varphi_+ \neq 0$ — скалярная функция из $L_p^+(\Gamma)$, не принадлежат $\operatorname{Im} \mathscr{P}\mathscr{A}_n$ $(n = 1, 2, \ldots)$. В самом деле, допустим, что

$$\varphi_+(\zeta)x = A_n(\zeta) f_+(\zeta) + f_-(\zeta) \qquad (f_+ \in \mathscr{L}_p^{\pm}(\Gamma, \mathfrak{B})).$$

Тогда

$$\varphi_+(\zeta) = \langle \varphi_+(\zeta)x, a\rangle = \langle A_n(\zeta) f_+(\zeta), a\rangle + \langle f_-(\zeta), a\rangle.$$

Так как $\langle A_n(\zeta) f_+(\zeta), a\rangle = \langle A(\zeta_0) f_+(\zeta), a\rangle = 0$ для $\zeta \in U_n$, то отсюда следует

$$\varphi_+(\zeta) = \langle f(\zeta), a\rangle \qquad (\zeta \in U_n),$$

что невозможно, так как $\langle f_-(\zeta), a\rangle \in L_p^-(\Gamma)$.

Этим доказано, что операторы $\mathscr{P}\mathscr{A}_n$ имеют бесконечномерные коядра, что в силу (6. 9) означает, что оператор $\mathscr{P}\mathscr{A}$, вопреки предположению, не является Φ-оператором.

Теорема доказана.

Отметим, что из теорем 5. 1 и 6. 1 вытекают теоремы 0. 2 и 0. 4, сформулированные во введении к первой части статьи. В самом деле, как отмечено в § 4, п. 1 этого параграфа, для любого банахова пространства $\mathfrak{B}$ алгебра $\mathfrak{W}(L(\mathfrak{B}))$ распадается и обладает свойствами а), б) и в). Пространство $\mathfrak{W}(\mathfrak{B})$, очевидно, является $\mathfrak{W}(L(\mathfrak{B}))$-пространством, откуда в силу теоремы 5. 1 следует теорема 0. 4. Как отмечено в § 4 п. 4 в случае гильбертова пространства, $\mathfrak{B}=\mathfrak{H}$ пространство $\mathscr{L}_2(\Gamma, \mathfrak{H})$ является распадающимся $\mathfrak{W}(L(\mathfrak{H}))$-пространтвом, откуда в силу теорем 5. 1 и 6. 1 вытекает теорема 0. 2.

§ 7. Еще две общие теоремы

1. Теорема 2. 1 не является следствием теоремы 5. 1, так как алгебра $C_\omega(K, L(\mathfrak{B}))$ не обладает свойствами б) и в). Приведем одно обобщение теоремы 5. 1, которое содержит также теорема 2. 1.

Пусть $M \supseteqq \Gamma$ — некоторое замкнутое множество, для которого существует последовательность $K_1 \supseteqq K_2 \supseteqq \ldots$ Γ-колец, такая, что

$$M = \bigcap_{j=1}^{\infty} K_j,$$

и пусть $\mathfrak{E}_M = \mathfrak{E}(M, L(\mathfrak{B}))$ — распадающаяся[6]) банахова алгебра непрерывных оператор-функций $A: M \to L(\mathfrak{B})$, которая обладает следующими свойствами:

а′) *Для любой оператор-функции $A \in \mathfrak{E}_M$ имеет место*

$$\max_{\zeta \in M} \|A(\zeta)\|_{\mathfrak{B}} \leqq C \|A\|_{\mathfrak{E}},$$

где константа C не зависит A.

б′) *Если все значения $A(\zeta)$ $(\zeta \in M)$ оператор-функции $A \in \mathfrak{E}_M$ принадлежат $GL(\mathfrak{B})$, то $A^{-1} \in \mathfrak{E}_M$.*

в′) *Все голоморфные оператор-функции $A: M \to L(\mathfrak{B})$ принадлежат алгебре $\mathfrak{E}_M$ и образуют в ней плотное множество.*

Пусть $\mathscr{B}$ — некоторое линейное подмножество пространства $\mathscr{L}_1$ которое со своей нормой является банаховым пространством. Пространство $\mathscr{B}$ будем называть $\mathfrak{E}_M$-пространством, если оно является распадающимся и обладает следующими свойствами:

α') *Для любой оператор-функции $A \in \mathfrak{E}_M$ имеет место $\mathscr{A}(\mathscr{B}) \leqq \mathscr{B}$ и сужение оператора $\mathscr{A}$ на $\mathscr{B}$ является ограниченным оператором в $\mathscr{B}$.*

β') *Все голоморфные вектор-функции $f: M \to \mathfrak{B}$ принадлежат $\mathscr{B}$ и для любого Γ-кольца $K \geqq M$ норма пространства $C_\omega(K, \mathfrak{B})$ сильнее нормы $\mathscr{B}$.*[7])

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[6]) В смысле определения из § 4, п. I.

[7]) Кроме того, предполагается что норма в пространстве $\mathscr{B}$ сильнее чем в $\mathscr{L}_1$.

Мы не останавливаемся здесь на примерах $\mathfrak{C}_M$-пространств, которые аналогичны примерам $\mathfrak{C}$-пространств, приведенным в § 4. Отметим лишь, что для всякой распадающейся алгебры, в которой норма слабее нормы пространства $C_\omega(K, \mathfrak{B})$, где K—произвольное Γ-кольцо, содержащее M, существует специальное $\mathfrak{C}_M$-пространство. Это пространство строится так же, как $\mathfrak{C}$-пространство $\mathscr{B}(\mathfrak{C})$ (см. § 4, п. 7), и будет обозначаться также через $\mathscr{B}(\mathfrak{C}_M)$.

Теорема 7. 1. *Пусть $\mathfrak{C}_M$-распадающаяся банахова алгебра, обладающая свойствами* а′), б′) *и* в′) *и $\mathscr{B}$—некоторое распадающееся $\mathfrak{C}_M$-пространство.*

Для того чтобы оператор-функция $A: M \to GL(\mathfrak{B})$ из $\mathfrak{C}_M$ допускала факторизацию в $\mathfrak{C}_M$, необходимо и достаточно, чтобы оператор $\mathscr{P}\mathscr{A}$ был Φ-оператором в пространстве $\mathscr{B}^+$.

Легко проверяется, что доказательство теоремы 5. 1 остается в силе и для теоремы 7. 1.

Очевидно, обе теоремы 2. 1 и 5. 1 являются частными случаями теоремы 7. 1.

Приведем одно следствие из теоремы 7. 1.

Следствие 7. 1. *Пусть пространство $\mathscr{B}$ распадается и обладает следующими свойствами:*

α'') *Для любой голоморфной оператор-функции $A: \Gamma \to L(\mathfrak{B})$ имеет место $\mathscr{A}(\mathscr{B}) \subseteq \mathscr{B}$ и сужение оператора $\mathscr{A}$ на $\mathscr{B}$ является ограниченным оператором в пространстве $\mathscr{B}$.*

β'') *Все голоморфные вектор-функции $f: \Gamma \to \mathfrak{B}$ принадлежат пространству $\mathscr{B}$ и для каждого Γ-кольца K норма пространства $C_\omega(K, \mathfrak{B})$ сильнее нормы $\mathscr{B}$.*[8])

Голоморфная оператор-функция $A: \Gamma \to GL(\mathfrak{B})$ допускает факторизацию относительно контура Γ в том только том случае, когда оператор $\mathscr{P}\mathscr{A}$ является Φ-оператором в $\mathscr{B}^+$.

2. Сделаем два замечания к общим теоремам. Пусть пространство $\mathscr{B}$ распадается и $A: \Gamma \to GL(\mathfrak{B})$ — любая непрерывная оператор-функция, для которой $\mathscr{A}(\mathscr{B}) \subseteq \mathscr{B}$, $\mathscr{A}^{-1}(\mathscr{B}) \subseteq \mathscr{B}$ и сужения операторов $\mathscr{A}$ и $\mathscr{A}^{-1}$ на $\mathscr{B}$ являются ограниченными операторами в $\mathscr{B}$.

Тогда легко проверить, что имеет место равенство

$$\mathscr{Q}+\mathscr{P}\mathscr{A}\mathscr{P} = \mathscr{A}(\mathscr{P}+\mathscr{Q}\mathscr{A}^{-1}\mathscr{Q})(\mathscr{I}+\mathscr{P}\mathscr{A}^{-1}\mathscr{Q})(\mathscr{I}-\mathscr{Q}\mathscr{A}\mathscr{P}), \tag{7. 1}$$

где $\mathscr{Q} = \mathscr{I}-\mathscr{P}$. Так как операторы $\mathscr{A}$, $\mathscr{I}+\mathscr{P}\mathscr{A}^{-1}\mathscr{Q}$ и $\mathscr{I}-\mathscr{Q}\mathscr{A}\mathscr{P}$ обратимы, то отсюда следует, что оператор $\mathscr{P}\mathscr{A}$ является Φ-оператором в $\mathscr{B}^+$ в том и только том случае, когда оператор $\mathscr{Q}\mathscr{A}^{-1}$ является Φ-оператором в $\mathscr{B}^-$. При этом

$$\dim \operatorname{Ker} \mathscr{P}\mathscr{A} = \dim \operatorname{Ker} \mathscr{Q}\mathscr{A}^{-1} \quad \text{и} \quad \dim \operatorname{Coker} \mathscr{P}\mathscr{A} = \dim \operatorname{Coker} \mathscr{Q}\mathscr{A}^{-1}.$$

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[8]) Кроме того, предполагается что норма в пространстве $\mathscr{B}$ сильнее чем в $\mathscr{L}_1$.

Следовательно, в формулировках теорем 0. 2, 0. 4, 5. 1, 6. 1 и 7. 1 и в следствии 7. 1 оператор $\mathcal{P}\mathcal{A}$ можно заменить оператором $\mathcal{Q}\mathcal{A}^{-1}$. Отметим еще, что если в предположениях одного из этих предложений оператор-функция A допускает факторизацию $A=A_-DA_+$ относительно Γ, то, как легко видеть, имеют место равенства

$$\dim \operatorname{Ker} \mathcal{P}\mathcal{A} = \dim \operatorname{Ker} \mathcal{Q}\mathcal{A}^{-1} = \dim \operatorname{Ker} \mathcal{P}\mathcal{D} = \sum_{\varkappa_j<0} \varkappa_j \tag{7. 2}$$

и

$$\dim \operatorname{Coker} \mathcal{P}\mathcal{A} = \dim \operatorname{Coker} \mathcal{Q}\mathcal{A}^{-1} = \dim \operatorname{Coker} \mathcal{P}\mathcal{D} = \sum_{\varkappa_j>0} \varkappa_j, \tag{7. 3}$$

где $\varkappa_j$-все частные индексы оператор-функции A.

3. В этом пункте приводится еще одна общая теорема. Ее формулировке предпошлем два определения.

Пусть $\mathfrak{E}_M$-некоторая банахова алгебра непрерывных оператор-функций $A: M\to L(\mathfrak{B})$. Подпространство $\mathfrak{N}\subseteq\mathfrak{E}_M$ назовем *конечно-порожденным*, если существует конечное число оператор-функций $X_j\in\mathfrak{N}$ и одномерных проекторов P_j $(j=1, 2, \dots, n)$ из $L(\mathfrak{B})$, таких, что $\mathfrak{N}$ совпадает с множеством всех оператор-функций вида

$$\sum_{j=1}^{n} X_j(\zeta)P_jA_j, \tag{7. 4}$$

где A_j — произвольные операторы из $L(\mathfrak{B})$. Линейный ограниченный оператор T, действующий в пространстве $\mathfrak{E}_M^+$, назовем *обобщенным Ф-оператором*, если он обладает следующими свойствами:

1) *оператор T — нормально разрешим, т. е. его множество значений замкнуто*;

2) *подпространство* $\operatorname{Ker} T$ *является конечнопорожденным*;

3) *к подпространству* $\operatorname{Im} T$ *существует прямое дополнение в* $\mathfrak{E}_M$, *являющееся конечнопорожденным подпространством.*

Теорема 7. 2. *Пусть $\mathfrak{E}_M$ — распадающаяся банахова алгебра, обладающая свойствами* а), б′) *и* в′), *и пусть оператор-функция $A: M\to GL(\mathfrak{B})$ принадлежит* $\mathfrak{E}_M$.

Для того чтобы оператор-функция A допускала факторизацию в $\mathfrak{E}_M$, необходимо и достаточно, чтобы оператор $\mathcal{P}\mathcal{A}$ был обобщенным Ф-оператором в пространстве $\mathfrak{E}_M^+$.

Доказательство. Пусть оператор-функция $A\in\mathfrak{E}_M$ допускает факторизацию в $\mathfrak{E}_M$:

$$A=A_-DA_+$$

где

$$D(\zeta) \sum_{j=1}^{n} \zeta^{\varkappa_j}P_j+P_0.$$

Оператор $\mathcal{PA}=\mathcal{PA}_-\mathcal{PDPA}_+$, причем $\mathcal{PA}_-$ и $\mathcal{PA}_+$ обратимы в $\mathfrak{E}_M^+$. Оператор $\mathcal{PD}$ распадается в прямую сумму конечного числа односторонне обратимых операторов. Следовательно, оператор $\mathcal{PD}$ и вместе с ним оператор $\mathcal{PA}$ являютя нормально разрешимыми.

Без труда проверяется, что подпространство Ker $\mathcal{PA}$ состоит из всех оператор-функций из $\mathfrak{E}_M$ вида

$$\sum_{\varkappa_j<0}\sum_{k=0}^{-\varkappa_j-1}\zeta^k A_+^{-1}(\zeta)P_jA_j,$$

где A_j — произвольные операторы из $L(\mathfrak{B})$.

Подпространство всех оператор-функций вида

$$\sum_{\varkappa_j>0}\sum_{j=0}^{\varkappa_j-1}\big(\mathcal{P}\zeta^{\varkappa_j}A_-(\zeta)\big)_0 P_jA_j,$$

где A_j пробегает все $L(\mathfrak{B})$, является прямым дополнением к Im $\mathcal{PA}$ в $\mathfrak{E}_M^+$. Следовательно, оператор $\mathcal{PA}$ является обобщенным Φ-оператором в $\mathfrak{E}_M^+$.

Пусть $\mathcal{PA}$ является обобщенным Φ-оператором в пространстве $\mathfrak{E}_M^+$. Образуем банахово пространство $\mathcal{B}(\mathfrak{E}_M)$, которое можно интерпретировать как подпространство $\mathfrak{E}_M$ всех вектор-функций вида $X(\zeta)R$, где $X(\zeta)$ пробегает $\mathfrak{E}_M$, а R-фиксированный одномерный проектор из $L(\mathfrak{B})$. Пространство $\mathcal{B}(\mathfrak{E}_M)$ является распадающимся $\mathfrak{E}_M$-пространством, причем

$$\mathcal{B}^+(\mathfrak{E}_M)=\{X_+(\zeta)R: X_+(\zeta)\in\mathfrak{E}_M^+\}.$$

Покажем, что оператор $\mathcal{PA}|\mathcal{B}^+(\mathfrak{E}_M)$ является Φ-оператором. В самом деле, пусть уравнение

$$\mathcal{P}(\mathcal{A}X)=YR,$$

где $Y\in\mathfrak{E}_M^+$, разрешимо в $\mathfrak{E}_M^+$ и $X\in\mathfrak{E}_M^+$ является его решением, тогда, очевидно, оператор-функция $X(\zeta)R$ также является решением этого уравнения. Следовательно,

$$\operatorname{Im}\mathcal{PA}|\mathcal{B}^+(\mathfrak{E}_M)=(\operatorname{Im}\mathcal{PA}|\mathfrak{E}_M^+)\cap\mathcal{B}^+(\mathfrak{E}_M).$$

Таким образом, оператор $\mathcal{PA}|\mathcal{B}^+(\mathfrak{E}_M)$ является нормально разрешимым.

Пусть ядро оператора $\mathcal{PA}$ в $\mathfrak{E}_M^+$ систоит из всех оператор-функций вида (7.4), тогда легко видеть, что подпространство Ker $(\mathcal{PA}|\mathcal{B}^+(\mathfrak{E})_M$ состоит из оператор-функций вида

(7.5) $$\sum_{j=1}^{n}X_j(\zeta)P_jA_jR.$$

Оно, очевидно, пробегает конечномерное подпространство.

Пусть некоторое прямое дополнение к подпространству $\operatorname{Im} \mathscr{P}\mathscr{A}$ в $\mathfrak{E}_M$ состоит из всех оператор-функций вида (7. 4), тогда легко видеть, что все оператор-функций вида (7. 5) образуют прямое дополнение к $\operatorname{Im}(\mathscr{P}\mathscr{A}|\mathscr{B}^+(\mathfrak{E}_M))$ в $\mathscr{B}^+(\mathfrak{E}_M)$. Следовательно, $\dim \operatorname{Coker}(\mathscr{P}\mathscr{A}|\mathscr{B}^+(\mathfrak{E}_M))<\infty$.

Таким образом, оператор $\mathscr{P}\mathscr{A}|\mathscr{B}^+(\mathfrak{E}_M)$ является Φ-оператором. В силу теоремы 7. 1 оператор-функция A допускает факторизацию в $\mathfrak{E}_M$.

Теорема доказана.

§ 8. Оператор-функции, удовлетворяющие условию Гельдера

В этом параграфе дополнительно предполагается, что контур Γ-гладкий.

Обозначим через $H_\alpha = H_\alpha(\Gamma, L(\mathfrak{B}))$, где α — фиксированное число, $0<\alpha<1$, банахову алгебру оператор-функций $A:\Gamma\to L(\mathfrak{B})$, удовлетворяющих условию Гельдера с показателем α. Норма в H_α вводится равенством

$$\|A\|_\alpha = \max_{\zeta\in\Gamma}\|A(\zeta)\|_{\mathfrak{B}} + \sup_{\zeta_1,\zeta_2\in\Gamma,\zeta_1\neq\zeta_2}|\zeta_1-\zeta_2|^{-\alpha}\|A(\zeta_1)-A(\zeta_2)\|_{\mathfrak{B}}.$$

Как уже отмечалось, алгебра H_α удовлетворяет условиям а) и б) из § 4, п. 1 и является распадающейся: $H_\alpha = H_\alpha^+ \dotplus (H_\alpha^-)_0$. В этом параграфе устанавливается следующая теорема.

Теорема 8. 1. *Пусть $\mathscr{B}$ — некоторое распадающееся H_α-пространство (определение см. § 4, п. 3). Оператор-функция $A:\Gamma\to GL(\mathfrak{B})$ из H_α допускает факторизацию в H_α в том и только том случае, когда оператор $\mathscr{P}\mathscr{A}$ является Φ-оператором в пространстве $\mathscr{B}^+$.*

Эта теорема не укладывается в общую схему теоремы 5. 1, так как алгебра H_α не обладает свойством в) из § 4, п. 1. Голоморфные оператор-функции $A:\Gamma\to L(\mathfrak{B})$ принадлежат H_α, однако не образуют в H_α плотное множество.

Доказательство теоремы 8. 1 основывается на трех леммах. Первая из них заменяет условие в) более слабым, вторая заменяет лемму 4. 1 более сильной, а третья по существу совпадает с леммой 4. 3 для алгебр $H_\alpha(\Gamma, L(\mathfrak{B}))$.

Лемма 8. 1. *Пусть $0<\beta<\alpha<1$. Тогда все функции вида*

$$\sum_{j=1}^{n} r_j(\zeta)X_j,$$

где $X_j\in L(\mathfrak{B})$ и r_j-рациональные функции с полюсами вне контура Γ, образуют плотное множество в H_α по норме H_β.

В скалярном случае эту лемму установил Г. Ф. Манджавидзе [9] (см. также [10], стр. 443). В случае, когда Γ является окружностью и $\mathfrak{B}$-гильбертовым пространством, эта лемма доказана в [11]. Приведенное там доказательство сохраняет силу и для рассматриваемого здесь более общего случая.

Лемма 8. 2. *Пусть* $0<\beta<\alpha<1$. *Тогда существует число* $\delta>0$ *такое, что каждая оператор-функция* $A\in H_\alpha$, *удовлетворяющая условию* $\|A-I\|_\beta<\delta$, *допускает каноническую факторизацию в* H_α.

Эта лемма в более общем и более абстрактном виде доказана в статье [12]. Она является обобщением одного предложения, установленного Г. Ф. Манджавидзе [9] (см. также [10], стр. 442—446) для матриц-функций, удовлетворяющих условию Гельдера.

Лемма 8. 3. *Для любой оператор-функции* $A:\Gamma\to GL(\mathfrak{B})$ *из* H_α *существуют оператор-функции* $E_\pm:\bar{F}^\pm\to GL(\mathfrak{B})$ *соответственно из* $H_\alpha^\pm$, *такие, что оператор-функция* E_-AE_+ *голоморфна на* Γ.

Доказательство. Пусть $0<\beta<\alpha$ и δ — константа из леммы 8. 2. В силу леммы 8. 1 оператор-функция A представима в виде $(I+M)G$, где $\|M\|_\beta<\delta$ и оператор-функция G голоморфна на Γ. Из леммы 8. 2 следует каноническая факторизация оператор-функции $I+M$ в H_α: $I+M = X_-X_+$. Снова применяя лемму 8. 1, получаем представление оператор-функции X_+G в виде $X_+G = H(I+N)$, где $\|N\|_\beta<\delta$, и оператор-функция H голоморфна на Γ. В силу леммы 8. 2 оператор-функция $I+N$ допускает каноническую факторизацию в H_α: $I+N = Y_-Y_+$. Так как оператор-функция HY_- голоморфна на Γ и $HY_- = X_-^{-1}AY_+^{-1}$, то этим лемма доказана.

Доказательство теоремы 8. 1 теперь совпадает с доказательством теоремы 5. 1, если вместо леммы 4. 3 воспользоваться леммой 8. 3.

В третьем параграфе было отмечено, что пространства $H_\alpha(\Gamma, \mathfrak{B})$ и $\mathscr{L}_2(\Gamma, \mathfrak{H})$ (в случае гильбертова пространства $\mathfrak{B}=\mathfrak{H}$) являются распадающимися H_α-пространствами. Таким образом, из теоремы 8. 1 следуют теоремы 0. 1 и 0. 3, сформулированные во введении.

Отметим, что в силу равенства (7. 1) в теоремах 0, 1. 0. 3 и 8. 1 оператор $\mathscr{P}\mathscr{A}$ можно заменить оператором $\mathscr{Q}\mathscr{A}^{-1}$. При этом если условия одной из этих теорем выполняются, то имеют место соотношения (7. 2) и (7. 3).

В заключение отметим, что сохраняет силу теорема 7. 2, если в ее формулировке заменить алгебру $\mathfrak{C}_M$ алгеброй $H_\alpha(\Gamma, L(\mathfrak{B}))$.

§ 9. Устойчивость

Пусть непрерывная оператор-функция $A:\Gamma\to GL(\mathfrak{B})$ допускает факторизацию относительно контура Γ:

$$A(\zeta)=A_-(\zeta)\left(P_0+\sum_{j=1}^{n}\zeta^{\varkappa_j}P_j\right)A_+(\zeta).$$

Число $\operatorname{Ind} A$ $(=\operatorname{Ind}(A,\Gamma))$, определямое равенством

$$\operatorname{Ind} A=\sum_{j=1}^{n}\varkappa_j.$$

называется *суммарным индексом* оператор-функции A относительно контура Γ. В статьях [7, 13] (см. также [14]) при дополнительных ограничениях на оператор-функции получены формулы для вычисления суммарного индекса и доказана его устойчивость при малых возмущениях. В этом параграфе теоремы об устойчивости распространяются на новые классы оператор-функций.

В этом параграфе $\mathfrak{F}$ обозначает либо распадающуюся алгебру $\mathfrak{C}(M, L(\mathfrak{B}))$ со свойствами а′), б′) и в′) (см. § 7), либо алгебру $H_\alpha(\Gamma, L(\mathfrak{B}))$ (см. § 8), в последнем случае предполагается, что контур Γ является гладким.

Отметим, что, в частности, $\mathfrak{F}$ может совпадать с алгеброй $C_\omega(K, L(\mathfrak{B}))$ (см. § 1), где K — некоторое Γ-кольцо, или с распадающейся алгеброй $\mathfrak{C}(\Gamma, L(\mathfrak{B}))$, удовлетворяющей условиям а), б) и в) (см. § 4, п. 1).

Через $\mathscr{B}$ обозначим некоторое распадающееся $\mathfrak{F}$-пространство (см. §§ 4 и 7) и через $\mathscr{P}$-проектор, проектирующий $\mathscr{B}$ на $\mathscr{B}^+$ параллельно $\mathscr{B}^-$.

Теорема 9. 1. *Пусть оператор-функция $A:\Gamma\to GL(\mathfrak{B})$ из алгебры $\mathfrak{F}$ допускает факторизацию в $\mathfrak{F}$. Тогда существует число $\delta=\delta(A)>0$ такое, что для любой оператор-функции $X\in\mathfrak{F}$, удовлетворяющей условию*

(9. 1) $$\|x\|_{\mathscr{B}}<\delta$$

оператор-функция $A+X$ также допускает факторизацию в $\mathfrak{F}$, причем

(9. 2) $$\operatorname{Ind}(A+X)=\operatorname{Ind} A \quad \text{и} \quad \sum_{\varkappa_j'>0}\varkappa_j'\leqq\sum_{\varkappa_j>0}\varkappa_j,$$

где $\varkappa_j$ — частные индексы оператор-функции A, а $\varkappa_j'$ — частные индексы оператор-функции $A+X$.

Доказательство. В силу теорем 7. 1 или 8. 1 в зависимости от выбора алгебры $\mathfrak{F}$ оператор $\mathscr{P}\mathscr{A}$ является Φ-оператором в пространстве $\mathscr{B}^+$, причем согласно (7. 2) и (7. 3)

$$\operatorname{Ind} A=-\operatorname{Ind}\mathscr{P}\mathscr{A}^{9)} \quad \text{и} \quad \sum_{\varkappa_j>0}\varkappa_j=\dim\operatorname{Coker}\mathscr{P}\mathscr{A}.$$

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[9]) Через $\operatorname{Ind}\mathscr{P}\mathscr{A}$ обозначается индекс оператора $\mathscr{P}\mathscr{A}$, т. е.

$$\operatorname{Ind}\mathscr{P}\mathscr{A}=\dim\operatorname{Ker}\mathscr{P}\mathscr{A}-\dim\operatorname{Coker}\mathscr{P}\mathscr{A}.$$

Из теорем об устойчивости Φ-операторов и их индексов [15] вытекает существование числа $\varepsilon>0$, такого, что для всех операторфункций $X\in\mathfrak{F}$, удовлетворяющих неравенству $\|\mathscr{P}\mathscr{X}\|_{\mathscr{B}}<\varepsilon$, оператор $\mathscr{P}\mathscr{A}+\mathscr{P}\mathscr{X}$ является Φ-оператором, причем

$$\operatorname{Ind}\mathscr{P}(\mathscr{A}+\mathscr{X}) = \operatorname{Ind}\mathscr{P}\mathscr{A} \text{ и } \dim\operatorname{Coker}\mathscr{P}(\mathscr{A}+\mathscr{X})\leqq\dim\operatorname{Coker}\mathscr{P}\mathscr{A}.$$

Осталось положить $\delta=\varepsilon\|\mathscr{P}\|^{-1}$ и воспользоваться снова одной из теорем 7. 1 или 8. 1 и равенствами (7. 2) и (7. 3).

Теорема доказана.

Отметим, что если пространство $\mathscr{B}$ совпадает с пространством $\mathscr{L}_p(\Gamma, \mathfrak{B})$ при некотором p, $1<p<\infty$, то в теореме 9. 1 условие (9. 1) принимает вид

$$\max_{\zeta\in\Gamma}\|X(\zeta)\|_{\mathfrak{B}}<\delta.$$

Это вытекает из того, что

$$\|X\|_{\mathscr{L}_p(\Gamma,\mathfrak{B})}=\max_{\zeta\in\Gamma}\|X(\zeta)\|_{\mathfrak{B}}.$$

§ 10. Одна теорема о неполной факторизации

В § 3 было введено следующее определение неполной факторизации. *Неполной факторизацией* оператор-функции $A:\Gamma\to GL(\mathfrak{B})$ называется ее представление в виде

$$A=\tilde{A}_-A_+,$$

где оператор-функция $A_+:\bar{F}^+\to GL(\mathfrak{B})$ непрерывна на $\bar{F}^+$ и голоморфна в F^+, а оператор-функция $\tilde{A}_-:\bar{F}^-\setminus\{\infty\}\to GL(\mathfrak{B})$ непрерывна на $\bar{F}^-\setminus\{\infty\}$ и голоморфна $F^-\setminus\{\infty\}$.

Теорема 3. 1 допускает следующее обобщение.

Теорема 10. 1. *Пусть $\mathfrak{F}$ обозначает такую же банахову алгебру, оператор-функций, как в § 9. Тогда любая оператор-функция $A\in\mathfrak{F}$, прининмающая значения только из $GL(\mathfrak{B})$, допускает неполную факторизацию: $A=\tilde{A}_-A_+$, причем $\tilde{A}_-^{\pm1}, A_+^{\pm1}\in\mathfrak{F}$.*

Эта теорема выводится из теоремы 3. 1 с помощью леммы 5. 3 или соответствующих предложений для $\mathfrak{C}_M$ и H_α.

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A generalization of the Halmos—Bram criterion for subnormality

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Introduction. In [3] and [1] HALMOS and BRAM show that a continuous linear operator A on a complex Hilbert space X is subnormal if and only if $\sum_{i,j=0}^{n} (A^i x_j, A^j x_i) \geqq 0$ for all finite collections $x_0, \ldots, x_n$ of X. In Section 1 we generalize this criterion by showing that A is subnormal if and only if $\sum_{i,j=0}^{n} (A^{i+j} x_j, A^{i+j} x_i) \geqq 0$ for all finite subcollections $x_0, \ldots, x_n$ of X. As an application of this criterion we show in Section 2 that an operator A is the restriction of a normal partial isometry to an invariant subspace if and only if $A = A^* A^2$ and $\|A\| \leqq 1$. In Section 3 we show, using our new criterion for subnormality, that an operator A is subnormal if and only if $\{A^{*n} A^n\}_{n=0}^{\infty}$ is a Hausdorff moment sequence.

Throughout the paper X is a complex Hilbert space with inner product $(\, ,\,)$ and norm $\|\ \|$. If B is a continuous linear operator on X, then B^* is the adjoint of B. B is *normal* if $BB^* = B^* B$, *quasi-normal* if $B(B^* B) = (B^* B)B$, an *isometry* if $B^* B = I$ and a *partial isometry* if $(B^* B)^2 = B^* B$. An operator A is *subnormal* if it is the restriction of a normal operator B to an invariant subspace of B and *hyponormal* if $AA^* \leqq A^* A$. A sequence $\{C_n\}_0^{\infty}$ of operators on X is a *Hausdorff moment sequence* if there exists a positive operator measure φ on some interval $[a, b]$ such that $C_n = \int_a^b t^n \, d\varphi$ for each nonnegative integer n.

1. A criterion for subnormality. The Halmos—Bram criterion that an operator A on X be subnormal is that $\sum_{i,j=0}^{n} (A^i x_j, A^j x_i) \geqq 0$ for all finite collections $x_0, \ldots, x_n$ of X. We generalize this as follows:

Theorem 1. *An operator A on a complex Hilbert space X is subnormal if and only if A satisfies*

$$(\mathrm{S}_1) \qquad \sum_{i,j=0}^{n} (A^{i+j} x_j, A^{i+j} x_i) \geqq 0$$

for each finite collection $x_0, \ldots, x_n$ of X.

Proof. The proof of the necessity of the condition is easy. Note that (S_1) is the special case of the Halmos—Bram condition which we obtain by choosing $x_i = A^i x_i'$ for $i = 0, \ldots, n$.

To prove the sufficiency of the condition we imitate the techniques of Halmos and Bram and prove that if A satisfies condition (S_1), then A is the restriction of a quasi-normal operator to an invariant subspace. This will complete our proof, since every quasi-normal operator is subnormal ([4, problem 154]).

Assume now that A satisfies condition (S_1). The first step in the proof will be to show that A also satisfies

$$(S_2) \qquad \sum_{i,j=0}^{n} (A^{i+j+1}x_j, A^{i+j+1}x_i) \leqq \|A\|^2 \sum_{i,j=0}^{n} (A^{i+j}x_j, A^{i+j}x_i).$$

To obtain this result we outline a modification of BRAM's proof in [1, Theorem 1, p. 76].

Let $\varepsilon > 0$ and let $A_1 = A/(\|A\| + \varepsilon)$. A_1 also satisfies condition (S_1). Let $Y = l^2(X)$. Define C on Y by $(Cy)_i = \sum_{j=0}^{\infty} A_1^{*i+j} A_1^{i+j} y_j$. An argument similar to that used by Bram shows that C is a well-defined, bounded operator on Y and that $C \geqq 0$ on Y. Now define B on Y by $(By)_i = A_1 y_i$. A computation almost identical to that used by Bram shows that $\|B^* CBy\| \leqq \|Cy\|$ for all y in Y and hence by [5, p. 426] that $B^* CB \leqq C$ since $\|B\| = \|A_1\| < 1$. It now follows that if $x_0, \ldots, x_n$ are elements of X, then

$$\sum_{i,j=0}^{n} (A^{i+j+1}x_j, A^{i+j+1}x_i) \leqq (\|A\| + \varepsilon)^2 \sum_{i,j=0}^{n} (A^{i+j}x_j, A^{i+j}x_i).$$

Since ε was an arbitrary positive number, condition (S_2) is satisfied.

The second step in the proof of the theorem is the construction of a quasi-normal extension of A. The following modification of HALMOS' proof in [3] will give us this result.

Let $\tilde{X}$ be the set of all sequences $\{x_i\}_{i=-\infty}^{\infty}$ in X such that $x_i = 0$ for $i < 0$ and $x_i \neq 0$ for at most a finite number of i. On $\tilde{X}$ define

$$(\tilde{x}, \tilde{y}) = \sum_{i,j} (A^{i+j}x_j, A^{i+j}y_i).$$

Let $\tilde{Y}$ be the set of equivalence classes obtained by identifying $\tilde{x}$ with 0 if $(\tilde{x}, \tilde{x}) = 0$. Then since A satisfies condition (S_1), $(\ ,\ )$ is an inner product on $\tilde{Y}$. Define D on

$\tilde{X}$ by $(D\tilde{x})_i = Ax_i$. Using the fact that A also satisfies condition (S_2), we have

$$(D\tilde{x}, D\tilde{x}) = \sum_{i,j} (A^{i+j+1}x_i, A^{i+j+1}x_j) \leqq \|A\|^2 \sum_{i,j} (A^{i+j}x_i, A^{i+j}x_j) = \|A\|^2 (\tilde{x}, \tilde{x}).$$

It now follows that D may be considered to be a continuous linear operator on $\tilde{Y}$.

Define E on $\tilde{X}$ by $(E\tilde{x})_i = x_{i-1}$ and note that $DE = ED$ on $\tilde{X}$. Furthermore on $\tilde{X}$ we have the relation

$$(D\tilde{x}, D\tilde{y}) = \sum_{i,j} (A^{i+j+1}x_i, A^{i+j+1}x_j) = \sum_{i,j} (A^{i+j}x_i, A^{i+j}x_{j-1}) = (\tilde{x}, E\tilde{y}).$$

Thus on the completion of $\tilde{Y}$, the extensions of D and E satisfy the equation $E = D^* D$. However, we have already observed that D commutes with E. Therefore the extension of D to the completion of $\tilde{Y}$ is a quasi-normal extension of A and the proof of the theorem is complete.

In [6, Theorem 7, p. 73] MAC NERNEY shows that a sequence $\{C_n\}_{n=0}^{\infty}$ of Hermitian operators on X is a Hausdorff sequence for the interval $[a, b]$ if and only if

$$a \sum_{i,j=0}^{n} (x_i, C_{i+j}x_j) \leqq \sum_{i,j=0}^{n} (x_i, C_{i+j+1}x_j) \leqq b \sum_{i,j=0}^{n} (x_i, C_{i+j}x_j)$$

for each finite collection $x_0, \ldots, x_n$ in X. Using this result and Theorem 1, we readily obtain the following:

Corollary. *An operator A on X is subnormal if and only if $\{A^{*n}A^n\}_{n=0}^{\infty}$ is a Hausdorff moment sequence.*

We note that if A is subnormal, then A is quasi-normal if and only if $A^{*n}A^n = \int t^n d\varphi$ where φ is a spectral measure (that is, φ is a projection-valued operator measure). The proof of this assertion is simple. If A is quasi-normal, then $A^{*n}A^n = (A^*A)^n$ for $n \geqq 0$ and thus $A^{*n}A^n = \int t^n d\varphi$ where φ is the spectral resolution of A^*A. Conversely, if $A^{*n}A^n = \int t^n d\varphi$ and φ is projection-valued, then $A^{*n}A^n = (A^*A)^n$ for $n \geqq 0$. Furthermore, by the last corollary A is subnormal and hence hyponormal. However if A is hyponormal and $A^{*2}A^2 = (A^*A)^2$, then $(A^*A - AA^*)A = 0$, proving that A is quasi-normal.

2. The operator equation $A = A^*A^2$. Consider the weighted shift A on l^2 defined by $A(x_0, x_1, \ldots) = (0, 2x_0, x_1, x_2, \ldots)$. A simple computation shows that $A = A^*A^2$. However, since the weights of A are not monotone increasing, A is not hyponormal [4, p. 160] and consequently not subnormal. Thus not every operator satisfying the equation $A = A^*A^2$ is subnormal. The additional hypothesis needed to force A to be subnormal is that $\|A\| \leqq 1$. We are now able to completely characterize operators satisfying these two conditions.

Theorem 2. *Let A be an operator on a complex Hilbert space X. A is subnormal and the minimal normal extension of A is a partial isometry if and only if $\|A\| \leqq 1$ and $A = A^*A^2$.*

Proof. Assume first that B is a normal partial isometry on a Hilbert space Y, containing X, and that $B = A$ on X. Since every partial isometry has norm $\leqq 1$, we have $\|A\| \leqq \|B\| \leqq 1$. Let P be the projection of Y onto X. Then for each x in X we have $A^*A^2x = PB^*B^2x = PBB^*Bx$ (since B is normal) $= PBx$ (since B is a partial isometry) $= Bx = Ax$ and consequently, $A = A^*A^2$.

Now assume that $A = A^*A^2$ and $\|A\| \leqq 1$. A simple inductive argument shows that $A^{*k}A^k = A^*A$ for each integer $k \geqq 1$. Therefore if $x_0, \ldots, x_n$ are elements of X,

$$\sum_{i,j=0}^{n} (A^{i+j}x_j, A^{i+j}x_i) = \sum_{i,j=0}^{n} (Ax_j, Ax_i) + \|x_0\|^2 - \|Ax_0\|^2 =$$
$$= \left\| \sum_{i=0}^{n} Ax_i \right\|^2 + \|x_0\|^2 - \|Ax_0\|^2 \geqq 0 \text{ since } \|A\| \leqq 1.$$

By Theorem 1 we know that A is subnormal. Let $B: Y \to Y$ be the minimal normal extension of A. It remains to show that B is a partial isometry. Let P be the projection of Y onto X. Then for x in X, $\|PB^*B^2x\| = \|A^*A^2x\| = \|Ax\| = \|A^3x\|$ (since $A^{*3}A^3 = A^*A) = \|B^3x\| = \|B^*B^2x\|$. Therefore $B^*B^2x \in X$ for all x and consequently $B^*B^2 = B$ on X. Since B is the minimal normal extension of A, the set of vectors $\left\{ \sum_{i=0}^{n} B^{*i}x_i : x_i \in X \right\}$ is dense in Y and consequently $B^*B^2 = B$ on a dense subset of Y. This is sufficient to imply that B is a partial isometry. The proof is complete.

The assertion in Theorem 2 parallels the assertion that an operator A is an isometry if and only if A is subnormal and the minimal normal extension of A is unitary.

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Über einseitige Approximation durch Polynome. III

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1. Einführung

In dieser Arbeit, die die Fortsetzung der Arbeiten [1] bis [4] bildet, werden wir den folgenden Satz beweisen.

Hauptsatz. *Es sei* $w_k(x)=e^{-x^{2k}}$ $(k=1, 2, \ldots)$. *Ferner sei* $F(x)$ *die r-fache* $(r=0, 1, \ldots)$ *Integralfunktion einer Funktion* $F^{(r)}(x)$, *die auf jedem endlichen Intervall von beschränkter Schwankung ist und der Bedingung*

$$\int_{-\infty}^{\infty} w_k(x)|dF^{(r)}(x)| < +\infty \tag{1}$$

genügt, ferner sei

$$|F(x)| < C_1(1+x^{2s}) \qquad (-\infty<x<\infty), \tag{2}$$

wobei s eine festgehaltene, nichtnegative ganze Zahl bezeichnet). Dann existieren für jede ganze Zahl* $n(>s)$ *Polynome* $p_n(x)$ *und* $P_n(x)$ *höchstens* $(2n-2)$*-ten Grades, mit denen die folgenden Relationen bestehen:*

$$p_n(x)\leqq F(x)\leqq P_n(x) \qquad (-\infty<x<\infty), \tag{3}$$

$$\int_{-\infty}^{\infty} [P_n(x)-p_n(x)]w_k(x)dx = O\{n^{-1-\left(\frac{1}{2k}\right)(r+1)}\}. \tag{4}$$

Wir bemerken, daß dieser Satz für $k=1$ in [2] bewiesen wurde.

Die Ergebnisse bezüglich der einseitigen Approximation auf $[0, \infty)$ der Arbeit [3] kann man ganz analog verallgemeinern. Die Resultate des 2. Teiles vorliegender Arbeit sind dabei in beiden Fällen wesentlich.

In dieser Arbeit wollen wir die Bezeichnungen des Buches [5] von G. FREUD gebrauchen. Also bedeuten $p_n(w; x)$ $(n=0, 1, \ldots)$ orthonormale Polynome mit der Gewichtsfunktion $w(x)$:

$$x_{1n}(w) > x_{2n}(w) > \cdots > x_{nn}(w)$$

* C_1 und später C_2, usw. bedeuten nichtnegative, von x und n unabhängige Konstanten.

sind die Nullstellen des Polynoms $p_n(w;x)$;

$$\psi_n(w;x,\xi) = p_{n-1}(w;\xi)p_n(w;x) - p_n(w;\xi)p_{n-1}(w;x)$$

ist das zu $w(x)$ gehörende, in der Veränderlichen x quasi-orthogonale Polynom; n^* bedeutet den Grad von $\psi_n(w;x,\xi)$,

$$\xi_1 = \xi_{1n}(w;\xi) > \xi_2 > \dots > \xi_{n^*}$$

sind die Nullstellen von $\psi_n(w;x,\xi)$ (wir bemerken, daß $\xi \in \{\xi_i\}_{i=1}^{n^*}$ und zwar im folgenden immer $\xi = \xi_\sigma$); $\lambda_n(w;\xi)$ bezeichne die Christoffelsche Funktion des zu $w(x)$ gehörenden Gauß—Jacobischen Quadraturverfahrens. Mit $\mathbf{P}_n$ wollen wir die Menge der Polynome höchstens n-ten Grades bezeichnen.

2. Die Abschätzung der Christoffelschen Funktionen und der Entfernungen zwischen den Nullstellen der quasiorthogonalen Polynome

In diesem Paragraphen werden wir von den folgenden bekannten Ergebnissen Gebrauch machen.

Hilfssatz A (vgl. [5] Satz I. 4. 1). *Bezeichnet $w(x)$ eine beliebige Gewichtsfunktion, so gilt*

$$\lambda_{n+1}(w;\xi) = \min_{\substack{\pi_n \in \mathbf{P}_n \\ \pi_n(\xi) \neq 0}} \pi_n^{-2}(\xi) \int_{-\infty}^{\infty} \pi_n^2(x) w(x)\,dx \qquad (n=0,1,\dots). \tag{5}$$

Hilfssatz B (vgl. G. Freud [6], Satz 2). *Für jedes Polynom $\pi_n \in \mathbf{P}_n$ gilt*

$$\int_{-\infty}^{\infty} \pi_n^2(x) w_k(x)\,dx \leqq C_2 \int_{-C_3 n^{\frac{1}{2k}}}^{C_3 n^{\frac{1}{2k}}} \pi_n^2(x) w_k(x)\,dx. \tag{6}$$

Hilfssatz C (vgl. G. Freud [6], Hilfssatz 1). *Zu jeder natürlichen Zahl n gibt es ein Polynom $\varrho_n \in \mathbf{P}_n$, für welches die Ungleichungen*

$$0 < \varrho_n(x) < e^{\frac{x^k}{2}} \qquad (0 \leqq x < \infty) \tag{7}$$

und

$$\varrho_n(x) > \frac{1}{2} e^{\frac{x^k}{2}} \qquad (0 \leqq x \leqq c_4 n^{\frac{1}{2k}}) \tag{8}$$

erfüllt sind.

Hilfssatz D (vgl. [5], Satz III. 3. 2). *Es sei* $v(x)=x^{-\frac{1}{2}}$ *für* $0<x<1$ *und* $v(x)=0$ *sonst. Dann gilt*

(9) $$\lambda_{n+1}(v;\xi)<C_5 n^{-1} \qquad (0\leqq\xi\leqq 1).$$

Auf Grund dieser Ergebnisse läßt sich der folgende Satz einfach beweisen. Es sei $u_k(x)=\frac{1}{\sqrt{x}}w_k(\sqrt{x})$ $(x>0)$.

Satz 1. *Für eine beliebige natürliche Zahl n gilt*

(10) $$\lambda_{n+1}(u_k;\xi)<C_6 n^{-1+\frac{1}{2k}}w_k(\sqrt{\xi}) \qquad (0\leqq\xi<C_7 n^{\frac{1}{k}}).$$

Beweis. Infolge von (5) und (6) besteht die Ungleichung

$$\lambda_{n+1}(u_k;\xi)<C_8 \min_{\substack{\pi_n\in\mathbf{P}_n\\ \pi_n(\xi)\neq 0}} \pi_n^{-2}(\xi)\int_0^{C_9 n^{\frac{1}{k}}}\pi_n^2(x)u_k(x)\,dx.$$

Nun sei $\pi_n=\varrho_{\left[\frac{n}{2}\right]}\cdot\varphi_{\left[\frac{n}{2}\right]}$, wobei $\varphi_{\left[\frac{n}{2}\right]}$ ein beliebiges Polynom höchstens $\left[\frac{n}{2}\right]$-ten Grades bezeichnet. Infolge von (7) und (8) bestehen die folgenden Zusammenhänge:

$$\lambda_{n+1}(u_k;\xi)<C_{10}w_k(\sqrt{\xi})\min_{\substack{\varphi_{\left[\frac{n}{2}\right]}\in\mathbf{P}_{\left[\frac{n}{2}\right]}\\ \varphi_{\left[\frac{n}{2}\right]}(\xi)\neq 0}}\varphi_{\left[\frac{n}{2}\right]}^{-2}(\xi)\int_0^{C_9 n^{\frac{1}{k}}}\varphi_{\left[\frac{n}{2}\right]}^2(x)\frac{1}{\sqrt{x}}\,dx=$$

$$=C_{11}w_k(\sqrt{\xi})n^{\frac{1}{2k}}\min_{\substack{\varphi_{\left[\frac{n}{2}\right]}\in\mathbf{P}_{\left[\frac{n}{2}\right]}\\ \varphi_{\left[\frac{n}{2}\right]}(C_9^{-1}n^{-\frac{1}{k}}\xi)\neq 0}}\varphi_{\left[\frac{n}{2}\right]}^{-2}(C_9^{-1}n^{-\frac{1}{k}}\xi)\int_0^1\varphi_{\left[\frac{n}{2}\right]}^2(x)\frac{1}{\sqrt{x}}\,dx.$$

Der letzte Ausdruck ist auf Grund der Formel (5) gleich

$$C_{11}w_k(\sqrt{\xi})n^{\frac{1}{2k}}\lambda_{\left[\frac{n}{2}\right]+1}(v;C_9^{-1}n^{-\frac{1}{k}}\xi).$$

Satz 1 folgt unmittelbar aus (9).

Folgerung 1. *Für jedes n gilt die Ungleichung*

(11) $$\lambda_{n+1}(w_k;\xi)<C_{12}n^{-1+\frac{1}{2k}}w_k(\xi) \qquad (|\xi|<C_{13}n^{\frac{1}{2k}}).$$

Tatsächlich ist laut Hilfssatz A

$$\lambda_{n+1}(w_k;\xi)\leqq \min_{\substack{\pi_{[\frac{n}{2}]}\in P_{[\frac{n}{2}]}\\ \pi_{[\frac{n}{2}]}(\xi^2)\neq 0}} \pi_{[\frac{n}{2}]}^{-2}(\xi^2)\int_{-\infty}^{\infty}\pi_{[\frac{n}{2}]}^{2}(x^2)w_k(x)\,dx=\lambda_{[\frac{n}{2}]+1}(u_k;\xi^2)$$

(vgl. G. FREUD [6]; der Grundgedanke des Beweises von Satz 1 wurde ebenfalls aus dieser Arbeit entnommen).

Satz 2. *Bezeichne ξ' die größte der von ξ links liegenden Nullstellen des quasi-orthogonalen Polynoms $\psi_n(u_k;x,\xi)$. Falls $\xi'\geqq 0$ gilt, so besteht die Ungleichung*

$$\xi-\xi'<C_{14}n^{-1+\frac{1}{2k}}\sqrt{\xi}\quad \left(0<\xi<C_7 n^{\frac{1}{k}}\right). \tag{12}$$

Beweis. Infolge der Posseschen Ungleichungen (vgl. [5], I. (5. 10)) ist die Relation

$$\int_{\xi'}^{\xi} e^{x^k}u_k(x)\,dx\leqq e^{\xi'^k}\lambda_n(u_k;\xi')+e^{\xi^k}\lambda_n(u_k;\xi)$$

erfüllt. Wendet man auf die rechte Seite Satz 1 an, so gelant man zur Ungleichung

$$\left(\sqrt{\xi}-\sqrt{\xi'}\right)<C_6 n^{-1+\frac{1}{2k}}\quad \left(0<\xi<C_7 n^{\frac{1}{k}}\right). \tag{13}$$

Durch Multiplikation beider Seiten mit $\left(\sqrt{\xi}+\sqrt{\xi'}\right)$ ergibt sich (12).

Aus Satz 2 ergibt sich mit Hilfe der Theorie der Orthogonalpolynome (mit allgemeiner Gewichtsfunktion):

Folgerung 2. *Bezeichne ξ^* jene Nullstelle des quasi-orthogonalen Polynoms $\psi_n(w_k;x,\xi)$, die im Intervall $(-|\xi|,|\xi|)$ am nächsten zu ξ liegt; diese Nullstelle genügt der Ungleichung*

$$|\xi-\xi^*|<C_{15}n^{-1+\frac{1}{2k}}\quad \left(|\xi|<C_{16}n^{\frac{1}{2k}}\right), \tag{14}$$

wobei $C_{16}<C_{13}$ gilt.

3. Hilfssätze

Hilfssatz 1. *Für beliebige reelle Zahlen $x>0$, $\alpha\leqq 0$, $\beta\geqq 1$ und für $j=0,1,2,\ldots$ gilt*

$$I_j(x)=\int_{x}^{\infty}(y-x)^j y^{\alpha}e^{-y^{\beta}}\,dy\leqq \frac{j!}{(\beta x^{\beta-1})^{j+1}}\cdot x^{\alpha}e^{-x^{\beta}}. \tag{15}$$

Beweis. Wir führen eine Induktion bezüglich j durch. Offensichtlich bestehen die Zusammenhänge

$$I_0(x)=\int_x^\infty y^\alpha e^{-y^\beta}\,dy \leqq x^\alpha \frac{1}{\beta x^{\beta-1}}\int_x^\infty \beta y^{\beta-1}e^{-y^\beta}\,dy=\frac{1}{\beta x^{\beta-1}}x^\alpha e^{-x^\beta},$$

und falls $j\geqq 1$ ist, dann gilt

$$I_j(x)=j\int_x^\infty (y-x)^{j-1}I_0(y)\,dy\leqq j\int_x^\infty (y-x)^{j-1}\frac{1}{\beta y^{\beta-1}}y^\alpha e^{-y^\beta}\,dy\leqq$$

$$\leqq \frac{j}{\beta x^{\beta-1}}I_{j-1}(x).$$

Hilfssatz 2. *Falls die Bedingung*

$$\int_{-\infty}^{\infty} w_k(x)\,|dF^{(r)}(x)|<+\infty$$

erfüllt ist, so gilt die Ungleichung

$$(16)\qquad |F^{(j)}(x)|w_k(x)|x|^{(2k-1)(r-j)}<C_{17}\qquad (j=0,1,\dots,r;\ -\infty<x<\infty).$$

Beweis. Der Hilfssatz muß nur für große Absolutwerte von x bewiesen werden; für kleine Werte von x ist nämlich die Behauptung trivialerweise erfüllt. Falls $j=r$, so gelten die Beziehungen

$$|w_k(x)[F^{(r)}(x)-F^{(r)}(0)]|\leqq\left|\int_0^x w_k(y)\,|dF^{(r)}(y)|\right|\leqq\int_{-\infty}^{\infty} w_k(y)\,|dF^{(r)}(y)|<+\infty.$$

Weiß man, daß für x die Ungleichung

$$|F^{(j)}(x)|<C_{18}w_k^{-1}(x)x^{(1-2k)(r-j)}\qquad (x>0)$$

besteht, so folgt hieraus die Ungleichung

$$|F^{(j-1)}(x)-F^{(j-1)}(\omega)|\leqq C_{18}\int_\omega^x w_k^{-1}(y)y^{(1-2k)(r-j)}\,dy=$$

$$=C_{18}\int_\omega^x \frac{[w_k^{-1}(y)y^{(1-2k)(r-j+1)}]'}{[2k-(2k-1)(r-j+1)y^{-2k}]}\,dy\qquad (x>\omega>0).$$

Ist nun beispielweise $\omega=(r+1)^{1/2k}$, so ist im letzteren Integral der Nenner nicht kleiner als 1, also gilt die Ungleichung

$$|F^{(j-1)}(x)-F^{(j-1)}(\omega)|\leqq C_{18}\int_\omega^x [w_k(y)y^{(1-2k)(r-j+1)}]'\,dy\qquad (x>\omega>0),$$

woraus sich (16) für positive Werte von x bereits leicht ergibt. Für negative x-Werte verläuft der Beweis ganz analog.

4. Beweis des Hauptsatzes

Hilfssatz 3 (vgl. [4], Satz 1). *Es sei $w(x)$ eine beliebige, im Intervall (a, b) definierte Gewichtsfunktion. Ferner sei $f(x)$ die r-fache $(r=0, 1, \dots)$ Integralfunktion einer Funktion $f_r(x)$ mit beschränkter Schwankung in (a, b). Ist $n \geqq r+6$ und $f_r(x) \equiv$ $\equiv$*const. *außerhalb des Intervalls $(a_1, b_1) \subset (x_{n-\left[\frac{r}{2}\right]-4,n}(w), x_{\left[\frac{r}{2}\right]+4,n}(w))$, so gibt es Polynome $\pi_n(x)$ und $\pi_n^*(x)$ höchstens $(2n-2)$-ten Grades, für welche die Ungleichungen*

$$\pi_n(x) \leqq f(x) \leqq \pi_n^*(x) \qquad (a<x<b)$$

erfüllt sind und die Relation

$$\int_a^b [\pi_n^*(x)-\pi_n(x)]w(x)\,dx = O\Big\{\int_{a_1}^{b_1} \Phi(\xi)\,|df_r(\xi)|\Big\} \qquad (n = r+6, r+7, \dots)$$

besteht, wobei $\Phi(\xi)$ eine beliebige solche Majorante der Funktion

$$\varphi(\xi) = [\xi_{\sigma-\left[\frac{r}{2}\right]-1,n}(w;\xi)-\xi_{\sigma+\left[\frac{r}{2}\right]+1,n}(w;\xi)]^r \max_{\xi_{\sigma+\left[\frac{r}{2}\right]+1} \leqq t \leqq \xi_{\sigma-\left[\frac{r}{2}\right]-1}} \lambda_n(w;t)$$

bezeichnet, für welche das Integral auf der rechten Seite existiert.

Wir wenden uns nun dem Beweis des Hauptsatzes zu. Die folgenden Überlegungen sind für $n \geqq n_0$ richtig, wobei n_0 nicht von x abhängt. Besteht $n<n_0$, so ist die Behauptung infolge von (2) trivialerweise wahr. Wie es in [2] bemerkt wurde, genügt es die Betrachtung von Funktionen, die für $x<0$ verschwinden.

Es sei

$$\omega_n \in \left[\frac{1}{4} C_{16} n^{\frac{1}{2k}}, \frac{1}{2} C_{16} n^{\frac{1}{2k}}\right]$$

eine Stetigkeitsstelle der Funktion $F^{(r)}(x)$, und es bestehe

$$F(x) = \sum_{j=0}^{r} \frac{F^{(j)}(\omega_n)}{j!}(x-\omega_n)^j + F^*(x) + F^{**}(x), \tag{17}$$

wobei F^* und F^{**} die Bedingungen des Hauptsatzes erfüllen, ferner $F^*(x)$ für $x \geqq \omega_n$ und $F^{**}(x)$ für $x \leqq \omega_n$ verschwinden, und in jedem endlichen Intervall die Schwankung von $F^{*(r)}$ bzw. von $F^{**(r)}$ nicht größer ist, als die Schwankung von $F^{(r)}$.

Auf $F^*(x)$ läßt sich Hilfssatz 3 unmittelbar anwenden. Auf Grund dessen gibt es ein Paar $p_n^*, P_n^* \in \mathbf{P}_{2n-2}$ derart, daß die Ungleichungen

$$p_n^*(x) \leqq F^*(x) \leqq P_n^*(x) \qquad (-\infty<x<\infty)$$

bestehen; aus (11) und (14) erhält man ferner mit Hilfe einfacher Rechnungen den Zusammenhang

$$\int_{-\infty}^{\infty}[P_n^*(x)-p_n^*(x)]w_k(x)\,dx \leqq C_{19}n^{-\left(1-\frac{1}{2k}\right)(r+1)}\int_0^{\omega_n} w_k(\xi)\,|dF^{*(r)}(\xi)| \leqq$$

$$\leqq C_{19}n^{-\left(1-\frac{1}{2k}\right)(r+1)}\int_{-\infty}^{\infty} w_k(\xi)\,|dF^{(r)}(\xi)| = O\{n^{-\left(1-\frac{1}{2k}\right)(r+1)}\}.$$

Da $F^{**}(x)$ für $x<\omega_n$ verschwindet, ergibt sich aus (2) und (17) das Bestehen von

(18) $$|F^{**}(x)|<C_{20}\omega_n^{2s}\gamma_0(x,\omega_n)+C_{21}\gamma_{2s}(x,\omega_n)+$$

$$+\sum_{\nu=0}^{r}|F^{(\nu)}(\omega_n)|\,\gamma_\nu(x,\omega_n)\equiv T_r(x,\omega_n),$$

wobei

$$\gamma_j(x,\omega_n)=\begin{cases}0 & \text{für}\quad x<\omega_n\\ \dfrac{1}{j!}(x-\omega_n)^j & \text{für}\quad x\geqq\omega_n\end{cases}$$

($j=0,1,\ldots$). Da $\gamma_j(x,\omega_n)$ als Funktion von x die j-fache Integralfunktion der Funktion $\gamma_0(x,\omega_n)$ ist, und die letztere auf der ganzen reellen Achse von beschränkter Schwankung ist, läßt sich Hilfssatz 3 anwenden. Also gibt es ein $p_n(x,\omega_n,j)$ und ein $P_n(x,\omega_n,j)$, die in x Polynome höchstens $(2n-2)$-ten Grades sind und mit denen die Ungleichungen

$$p_n(x,\omega_n,j)\leqq\gamma_j(x,\omega_n)\leqq P_n(x,\omega_n,j)\qquad(-\infty<x<\infty)$$

und

$$\int_{-\infty}^{\infty}[P_n(x,\omega_n,j)-p_n(x,\omega_n,j)]w_k(x)\,dx\leqq$$

$$\leqq C_{22}n^{-\left(1-\frac{1}{2k}\right)(j+1)}\int_{\omega_n-0}^{\omega_n} w_k(\xi)\,|d\gamma_0(\xi,\omega_n)| = C_{22}n^{-\left(1-\frac{1}{2k}\right)(j+1)}w_k(\omega_n)$$

bestehen. Hierbei wurden die Abschätzungen (11) und (14) angewendet.

Nun sei

$$P_n^{**}(x)=C_{20}\omega_n^{2s}P_n(x,\omega_n,0)+C_{21}P_n(x,\omega_n,2s)+\sum_{\nu=0}^{r}|F^{(\nu)}(\omega_n)|\,P_n(x,\omega_n,\nu).$$

Es ist klar, daß

$$-P_n^{**}(x)\leqq F^{**}(x)\leqq P_n^{**}(x)\quad(-\infty<x<\infty)$$

und

$$\int_{-\infty}^{\infty} P_n^{**}(x)\,w_k(x)\,dx = \int_{-\infty}^{\infty} [P_n^{**}(x) - T_r(x,\omega_n)]\,w_k(x)\,dx + \int_{\omega_n}^{\infty} T_r(x,\omega_n)\,w_k(x)\,dx \leqq$$

$$\leqq C_{22}\,w_k(\omega_n)[C_{20}\,\omega_n^{2s}\,n^{-1+\frac{1}{2k}} + C_{21}\,n^{-\left(1+\frac{1}{2k}\right)(2s+1)} +$$

$$+ \sum_{\nu=0}^{r} |F^{(\nu)}(\omega_n)|\,n^{-\left(1-\frac{1}{2k}\right)(\nu+1)}] + \int_{\omega_n}^{\infty} T_r(x,\omega_n)\,w_k(x)\,dx.$$

richtig sind. Der Ausdruck auf der rechten Seite der Ungleichung ist infolge der Hilfssätze 1 und 2, der Relation (18) und der Definition von ω_n von der Größenordnung $0\{n^{-\left(1-\frac{1}{2k}\right)(r+1)}\}$. Den Hauptsatz selbst erhalten wir schließlich, indem wir

$$p_n(x) = \sum_{\nu=0}^{r} \frac{F^{(\nu)}(\omega_n)}{\nu!}(x-\omega_n)^\nu + p_n^{**}(x) - P_n^{**}(x)$$

und

$$P_n(x) = \sum_{\nu=0}^{r} \frac{F^{(\nu)}(\omega_n)}{\nu!}(x-\omega_n)^\nu + P_n^{**}(x) + P_n^{**}(x)$$

setzen.

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Über eine Klasse nichtlinearer Eigenwertprobleme

Von HEINZ LANGER in Dresden (DDR)

In der Arbeit [1] betrachtet R. E. L. TURNER die Schar

$$M(\mu) = A-\mu I-\mu^2 B_2-\cdots-\mu^N B_N$$

mit nichtnegativen Operatoren $A, B_2, \ldots, B_N \in \mathfrak{R}$[1] in einem seperablen Hilbertraum $\mathfrak{H}$. Für einen vollstetigen Operator A erweist sich das nichtnegative Spektrum der Schar M als diskret. Wählt man in jedem der zugehörigen Eigenräume eine Basis, so zeigt R. E. L. TURNER, daß das System aller dieser Eigenvektoren eine Rieszsche Basis von $\mathfrak{H}$ bildet, wenn nur eine der folgenden Bedingungen erfüllt ist:

a) $A \in \mathfrak{S}_p$ für $p<\frac{1}{2}$;

b) $A \in \mathfrak{S}_p$ für $p<\frac{2}{3}$ und $B_2, \ldots, B_N \in \mathfrak{S}_2$.

Unlängst wurden diese Bedingungen von A. S. MARKUS und G. I. RUSSU [2] abgeschwächt zu einer der folgenden:

c) $A \in \mathfrak{S}_p$ für ein $p<1$;

d) $A \in \mathfrak{S}_p$ für $p < m-1$ $(2 \leqq m \leqq N)$ und $B_2=\cdots=B_{m-1}=0$.

In der vorliegenden Note zeigen wir, daß für die Gültigkeit der obigen Aussage an Stelle von a), b), c) oder d) nur vorausgesetzt zu werden braucht, daß A vollstetig ist, und auch die Voraussetzungen über $B_2, \ldots, B_N$ abgeschwächt werden können.[2] Darüber hinaus läßt sich im allgemeineren Falle $A \in \mathfrak{R}$, $A \geqq 0$, aus unseren

[1] $\mathfrak{R}$ bezeichne den Ring aller beschränkten linearen, $\mathfrak{S}_\infty$ die Menge der vollstetigen Operatoren in $\mathfrak{H}$; $\mathfrak{S}_p$, $p>0$, sei die Menge aller $A \in \mathfrak{S}_\infty$ mit $\Sigma \lambda_j^{p/2}(A^*A)<\infty$, wobei sich die Summation über alle Eigenwerte von A^*A erstreckt, jeder entsprechend seiner Vielfachheit oft gezählt. Der Operator $A \in \mathfrak{R}$ heißt nichtnegativ (streng positiv), wenn für alle $x \in \mathfrak{H}$ gilt: $(Ax, x) \geqq 0$ $\left((Ax, x) \geqq \gamma\|x\|^2\right.$ für ein geeignetes $\gamma>0$), wir schreiben dafür $A \geqq 0$ $(A \gg 0)$.

[2] Neulich bewiesen auch V. I. MACAJEV und A. I. VIROZUB (erscheint in *Funkcional. Anal. i Priložen.*) die Gültigkeit der obigen Aussage unter solchen allgemeinen Bedingungen. Ihre Methode unterscheidet sich jedoch wesentlich von unserer, indem sie zum Beweis der Existenz einer Lösung $Y \in \mathfrak{R}$ mit den Eigenschaften 1)—3) aus Satz 2 ein Ergebnis von I. C. GOHBERG und J. LEITERER zur Faktorisierung von Operatorfunktionen benutzen.

Ergebnissen die Existenz einer „Spektralschar" für das nichtnegative Spektrum der Schar M herleiten, worauf wir an anderer Stelle eingehen werden.

Zum Beweis der obigen Aussage führen wir die Schar M durch eine Parametertransformation auf eine Schar L:

(1) $$L(\lambda) = \lambda^N I + \lambda^{N-1} D_{N-1} + \cdots + \lambda D_1 + D_0,$$

$D_0, \ldots, D_{N-1} \in \mathfrak{R}$, zurück, für die ein Intervall $[\lambda_1, \lambda_2]$ der reellen Achse mit

$$L(\lambda_1) \ll 0, \quad L(\lambda_2) \gg 0, \quad L'(\lambda) \gg 0 \quad \text{für} \quad \lambda \in [\lambda_1, \lambda_2]$$

existiert, und zeigen, daß es dann eine Lösung $Z \in \mathfrak{R}$ der Operatorengleichung

(2) $$L(Z) \equiv Z^N + D_{N-1} Z^{N-1} + \cdots + D_1 Z + D_0 = 0$$

gibt, die ähnlich zu einem selbstadjungierten Operator ist.[3] Diese Methode der Betrachtung der zur Schar (1) gehörigen Operatorengleichung wurde von M. G. Krein und dem Verfasser für gewisse quadratische Scharen in [3], [4] (siehe auch [5], [6]) entwickelt. Wesentliche Hilfsmittel für unsere Betrachtungen sind die von P. H. Müller [7] angegebene Linearisierung der Schar L sowie einige einfache Aussagen der Theorie linearer Operatoren in J-Räumen.

Herrn Professor M. G. Krein danke ich sehr dafür, daß er mich auf die Arbeit [1] aufmerksam und mir diese zugänglich gemacht hat, den Herren Dr. A. S. Markus und Dr. Ju. L. Šmul'jan danke ich für wertvolle Hinweise.

1. In diesem Abschnitt stellen wir einige einfache Begriffe aus der Theorie der linearen Operatoren in J-Räumen zusammen.

Es sei $\mathfrak{H}$ ein Hilbertraum für das Skalarprodukt (x, y) $(x, y \in \mathfrak{H})$, G ein beschränkter und beschränkt invertierbarer indefiniter (d.h., es gibt Elemente $x, y \in \mathfrak{H}$ mit $(Gx, x) > 0$ und $(Gy, y) < 0$) selbstadjungierter Operator in $\mathfrak{H}$. Wir definieren in $\mathfrak{H}$ ein indefinites, sog. G-Skalarprodukt $[x, y]$ durch die Gleichung

(3) $$[x, y] = (Gx, y) \quad (x, y \in \mathfrak{H}).$$

Dann ist $\mathfrak{H}$ ein J-Raum für dieses indefinite Skalarprodukt und ein geeignetes positiv definites Skalarprodukt, dessen Norm der Ausgangsnorm äquivalent ist. Ein Element $x \in \mathfrak{H}$ heißt G-positiv (G-nichtnegativ usw.), wenn $[x, x] > 0$ ($[x, x] \geqq 0$ usw.) gilt. Die Menge aller G-nichtnegativen Elemente von $\mathfrak{H}$ bezeichnen wir mit $\mathfrak{P}_+$.

Ein Teilraum, der außer $x = 0$ nur aus G-positiven (G-nichtnegativen) Elementen besteht, heiße G-positiv (G-nichtnegativ). Gilt für die Elemente x eines G-positiven Teilraumes $\mathfrak{L}$ sogar

$$[x, x] \geqq \gamma \|x\|^2$$

mit einem $\gamma > 0$, so nennen wir $\mathfrak{L}$ streng G-positiv.

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[3] D. h., es gibt einen Operator $S \gg 0$, so daß SZ selbstadjungiert ist.

Das G-orthogonale Komplement $\mathfrak{E}^{[\perp]}$ eines Teilraumes $\mathfrak{E} \subset \mathfrak{H}$ besteht definitionsgemäß aus allen $x \in \mathfrak{H}$ mit $[x, \mathfrak{E}] = \{0\}$.

Ein beschränkter linearer Operator A in $\mathfrak{H}$ heißt bekanntlich G-selbstadjungiert, wenn

$$[Ax, y] = [x, Ay] \quad (x, y \in \mathfrak{H})$$

gilt.

Wir benötigen die folgenden einfachen Aussagen:

1°. *Liegt die Spektralmenge σ des G-selbstadjungierten Operators A symmetrisch zur reellen Achse, so ist auch der zu σ gehörige Rieszsche Projektor G-selbstadjungiert* ([8]).

2°. *Ist der Wertebereich eines G-selbstadjungierten Projektors G-nichtnegativ, so ist er sogar streng G-positiv* ([9]), *bildet also einen Hilbertraum bezüglich des G-Skalarproduktes.*

Die Aussage 2° benutzen wir zum Beweis des

Lemma 1. *Es sei E_t, $0 \leqq t \leqq 1$, eine Familie G-selbstadjungierter Projektoren, die bezüglich der gleichmäßigen Operatorentopologie stetig vom Parameter t abhängt. Dann folgt aus $E_0 \mathfrak{H} \subset \mathfrak{P}_+$, daß $E_t \mathfrak{H}$ für alle $0 \leqq t \leqq 1$ streng G-positiv ist.*

Beweis. Die Menge

$$\tau = \{t : E_t \mathfrak{H} \subset \mathfrak{P}_+\}$$

ist offensichtlich abgeschlossen. Wäre $\tau \neq [0, 1]$, so gäbe es einen Punkt t_0 mit

$$t_0 = \inf ([0, 1] \setminus \tau).$$

Dann gilt auf Grund von 2° für ein geeignetes $\gamma > 0$:

$$[E_{t_0} x, E_{t_0} x] \geqq \gamma \|E_{t_0} x\|^2$$

Für alle t aus einer hinreichend kleinen Umgebung von t_0 gilt weiter $E_t E_{t_0} \mathfrak{H} = E_t \mathfrak{H}$. Wählen wir $\delta > 0$ so, daß $\gamma - (2\delta + \delta^2)\|G\| > 0$ ausfällt, dann folgt für alle $x \in \mathfrak{H}$ und t mit $\|E_t - E_{t_0}\| \leqq \delta$:

$$[E_t F_{t_0} x, E_t E_{t_0} x] \geqq [E_{t_0} x, E_{t_0} x] - 2\|E_t - E_{t_0}\| \, \|E_{t_0} x\|^2 \|G\| -$$
$$- \|E_t - E_{t_0}\|^2 \|E_{t_0} x\|^2 \|G\| \geqq \left(\gamma - (2\delta + \delta^2)\|G\|\right) \|E_{t_0} x\|^2,$$

im Widerspruch zu der Tatsache, daß in jeder Umgebung von t_0 ein t mit $E_t \mathfrak{H} \not\subset \mathfrak{P}_+$ liegt.

2. Wir betrachten zunächst allgemein die Schar L:

$$L(\lambda) = \lambda^N D_N + \lambda^{N-1} D_{N-1} + \cdots + \lambda D_1 + D_0$$

mit Operatoren $D_0, \ldots, D_N \in \mathfrak{R}$. Definitionsgemäß besteht das Spektrum $\sigma(L)$ der Schar L aus allen Punkten λ der komplexen Ebene, für die der Nullpunkt $z = 0$

ein Punkt des Spektrums des Operators $L(\lambda)$ ist; sein Komplement bildet die Resolventenmenge $\varrho(L)$.

Ist $z=0$ ein Eigenwert von $L(\lambda_0)$, d.h., besteht die Gleichung $L(\lambda_0)x^{(0)}=0$ mit einem Element $x^{(0)}\in\mathfrak{H}$, $x^{(0)}\neq 0$, so heißt λ_0 ein Eigenwert der Schar L und das Element $x^{(0)}$ ein Eigenelement zu diesem Eigenwert; der von allen solchen Eigenelementen gebildete Teilraum heißt der zu λ_0 gehörige Eigenraum. Die Menge der Eigenwerte der Schar L bezeichnen wir mit $\sigma_p(L)$.

Ist $\lambda_0\in\sigma_p(L)$ und genügen die Elemente $x^{(0)}\neq 0, x^{(1)}, \ldots, x^{(k)}$ den Gleichungen

$$L(\lambda_0)x^{(\varkappa)}+\frac{1}{1!}\frac{\partial L(\lambda_0)}{\partial\lambda_0}x^{(\varkappa-1)}+\ldots+\frac{1}{\varkappa!}\frac{\partial^\varkappa L(\lambda_0)}{\partial\lambda_0^\varkappa}x^{(0)}=0, \qquad \varkappa=0,1,\ldots,k,$$

so bilden sie definitionsgemäß eine Jordansche Kette zum Eigenwert λ_0.

Wir setzen jetzt $D_N=I$ voraus. Dann ordnet man ([7], vergl. auch [4], [10]) der Schar L den Operator

$$\mathbf{L}=\begin{pmatrix} -D_{N-1} & -D_{N-2} & \cdots & -D_1 & -D_0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix} \tag{4}$$

im Produktraum $\mathbf{\mathfrak{H}}=\mathfrak{H}^{(1)}\oplus\mathfrak{H}^{(2)}\oplus\cdots\oplus\mathfrak{H}^{(N)}$ mit $\mathfrak{H}^{(1)}=\mathfrak{H}^{(2)}=\cdots=\mathfrak{H}^{(N)}=\mathfrak{H}$ zu. Bekanntlich gilt dabei

$$\sigma(L)=\sigma(\mathbf{L}), \quad \sigma_p(L)=\sigma_p(\mathbf{L}), \tag{5}$$

und die zu einem Eigenwert λ_0 gehörenden Jordanschen Ketten von L und $\mathbf{L}$ entsprechen einander eineindeutig; insbesondere besteht zwischen den Eigenelementen $x^{(0)}\in\mathfrak{H}$ und $\mathbf{x}^{(0)}\in\mathbf{\mathfrak{H}}$ von L bzw. $\mathbf{L}$ der Zusammenhang

$$\mathbf{x}^{(0)}=\begin{pmatrix} \lambda_0^{N-1}x^{(0)} \\ \vdots \\ \lambda_0 x^{(0)} \\ x^{(0)} \end{pmatrix}.$$

Die Resolvente $(\mathbf{L}-\lambda\mathbf{I})^{-1}$ gestattet für $\lambda\in\varrho(\mathbf{L})=\varrho(L)$ die Matrixdarstellung

$$(\mathbf{L}-\lambda\mathbf{I})^{-1}=\big(R_{ij}(\lambda)\big)_{i,j=1,\ldots,N} \tag{6}$$

mit

$$R_{ij}(\lambda)=-\lambda^{N-i}L^{-1}(\lambda)\sum_{m=1}^{j}D_{N-m+1}\lambda^{j-m}+\ldots, \qquad i,j=1,2,\ldots,N, \tag{7}$$

wobei $D_N=I$ gesetzt wurde und die nicht aufgeschriebenen Summanden Polynome in λ sind.

Sind die Operatoren $D_0, D_1, \dots, D_{N-1}$ selbstadjungiert, so überzeugt man sich leicht davon, daß der Operator $\mathbf{L}$ für den selbstadjungierten und beschränkt invertierbaren Operator

$$\mathbf{G} = \begin{pmatrix} 0 & 0 & \dots & 0 & I \\ 0 & 0 & \dots & I & D_{N-1} \\ 0 & 0 & \cdot^{\cdot^{\cdot}} & D_{N-1} & D_{N-2} \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & I & D_{N-1} & \cdot^{\cdot^{\cdot}} & \\ I & D_{N-1} & D_{N-2} & \dots & D_1 \end{pmatrix}$$

$\mathbf{G}$-selbstadjungiert ist.

Setzen wir für ein reelles λ und $x \in \mathfrak{H}$

$$\mathbf{x} = \begin{pmatrix} \lambda^{N-1} x \\ \vdots \\ \lambda x \\ x \end{pmatrix},$$

so ergibt sich für das $\mathbf{G}$-Skalarprodukt (3) leicht

$$[\mathbf{x}, \mathbf{x}] = (L'(\lambda)x, x). \tag{8}$$

3. Neben der Schar L betrachten wir entsprechend wie in [4], [5], [6] die Operatorengleichung

$$L(Z) \equiv D_N Z^N + D_{N-1} Z^{N-1} + \dots + D_1 Z + D_0 = 0.$$

Ist $Z \in \mathfrak{R}$ eine Lösung dieser Gleichung, so ist offensichtlich jedes Eigenelement von Z auch ein Eigenelement von L zu demselben Eigenwert. Man kann zeigen, daß dies für Jordansche Ketten von Z ebenfals zutrifft. Außerdem sieht man leicht, daß auch jeder approximative Eigenwert von Z[4] zu $\sigma(L)$ gehört.

Es sei von jetzt an stets $D_N = I$ vorausgesetzt. Wie in [4], [5] erhalten wir eine Lösung Z von (2) mit Hilfe des folgenden

Lemma 2. *Ist* $\mathfrak{E}$ *ein invarianter Teilraum von* $\mathbf{L}$ *der Gestalt*

$$\mathfrak{E} = \left\{ \begin{pmatrix} Z^{(N-1)} x \\ \vdots \\ Z^{(1)} x \\ x \end{pmatrix} : x \in \mathfrak{H} \right\} \quad \text{mit} \quad Z^{(1)}, \dots, Z^{(N-1)} \in \mathfrak{R},$$

so gilt $Z^{(j)} = (Z^{(1)})^j$, $j = 1, 2, \dots, N-1$, *und der Operator* $Z = Z^{(1)}$ *bildet eine Lösung*

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[4] Die komplexe Zahl λ heißt ein approximativer Eigenwert von Z, wenn eine Folge (x_n) mit $\|x_n\| = 1$ und $Zx_n - \lambda x_n \to 0$ existiert.

der Gleichung (2). *Ist umgekehrt* $Z \in \mathfrak{R}$ *eine Lösung von* (2), *so läßt* $\mathbf{L}$ *den Teilraum*

$$\mathfrak{E}_Z = \left\{ \begin{pmatrix} Z^{N-1}x \\ \vdots \\ Zx \\ x \end{pmatrix} : x \in \mathfrak{H} \right\}$$

invariant; dabei gilt $\sigma(Z) = \sigma(\mathbf{L}|\mathfrak{E}_Z)$.

Der einfache Beweis der ersten Aussagen dieses Lemmas kann dem Leser überlassen werden. Zum Beweis der Aussage über das Spektrum beachten wir die Beziehung

$$\mathbf{L} \begin{pmatrix} Z^{N-1}x \\ \vdots \\ Zx \\ x \end{pmatrix} = \begin{pmatrix} Z^N x \\ \vdots \\ Z^2 x \\ Zx \end{pmatrix},$$

somit sind die Gleichungen

$$\mathbf{L} \begin{pmatrix} Z^{N-1}x \\ \vdots \\ Zx \\ x \end{pmatrix} = \begin{pmatrix} Z^{N-1}y \\ \vdots \\ Zy \\ y \end{pmatrix} \quad \text{und} \quad Zx = y$$

äquivalent, woraus leicht die Behauptung folgt.

Wir ergänzen diese Betrachtungen über die Wurzeln von (2) durch ein Lemma, das jedoch im folgenden nicht benötigt wird.

Jede Lösung Z von (2) erzeugt eine Darstellung von L in der Form $L(\lambda) = L_Z(\lambda)(\lambda I - Z)$ mit der Schar

$$L_Z(\lambda) = \sum_{j=0}^{N-1} \lambda^{N-j-1} \Big(\sum_{k=0}^{j} D_{N-k} Z^{j-k} \Big), \quad D_N = I,$$

vom Grade $N-1$. Weiter bezeichne $\bar{L}_Z$ die Schar $\bar{L}_Z(\lambda) = L_Z^*(\bar{\lambda})$.

Lemma 3. *Ist* $Z \in \mathfrak{R}$ *eine Lösung von* (2), *so ist der der Schar* $\bar{L}_Z$ *zugeordnete Operator* $\mathbf{L}_Z$ *im Raume* $\mathfrak{H}^{N-1}$ *der Einschränkung von* $\mathbf{L}$ *auf das* $\mathbf{G}$*-orthogonale Komplement* $\mathfrak{E}_Z^{[\perp]}$ *von* $\mathfrak{E}_Z$ *isomorph.*[5] *Insbesondere gilt also*

$$\sigma(\bar{L}_Z) = \sigma(\mathbf{L}|\mathfrak{E}_Z^{[\perp]}).$$

[5] D. h., es gibt eine eineindeutige und in beiden Richtungen stetige Abbildung von $\mathfrak{H}^{N-1}$ auf $\mathfrak{E}_Z$, die $\bar{\mathbf{L}}_Z$ in $\mathbf{L}|\mathfrak{E}_Z$ überführt.

Beweis. Der Teilraum $\mathfrak{E}_Z^{[\perp]}$ ist offensichtlich invariant bezüglich $\mathbf{L}$. Man überzeugt sich unmittelbar davon, daß er aus allen Elementen von $\mathfrak{H}$ der Form

$$\begin{pmatrix} -\sum_{j=1}^{N-1}\left(\sum_{k=0}^{j} Z^{*j-k}D_{N-k}\right)x_{N-j-1} \\ x_{N-2} \\ \vdots \\ x_0 \end{pmatrix}, \quad x_0, x_1, \dots, x_{N-2} \in \mathfrak{H},$$

besteht ($D_N = I$). Für zwei solche Elemente ist die Beziehung

$$\mathbf{L}\begin{pmatrix} \cdots \\ x_{N-2} \\ \vdots \\ x_0 \end{pmatrix} = \begin{pmatrix} \cdots \\ y_{N-2} \\ \vdots \\ y_0 \end{pmatrix}$$

äquivalent den folgenden $N-1$ Gleichungen:

$$\begin{aligned} -\sum_{j=1}^{N-1}\left(\sum_{k=0}^{j} Z^{*j-k}D_{N-k}\right)x_{N-j-1} &= y_{N-2}, \\ x_{N-2} &= y_{N-3}, \\ \vdots \quad & \quad \vdots \\ x_1 &= y_0. \end{aligned}$$

Daraus folgt leicht die Behauptung.

4. Hauptergebnis dieser Mitteilung ist der

Satz 1. *Die Schar L: $L(\lambda) = \lambda^N I + \lambda^{N-1}D_{N-1} + \cdots + \lambda D_1 + D_0$ genüge der folgenden Bedingung: Es existiert ein Intervall $[\lambda_1, \lambda_2]$ der reellen Achse mit $L(\lambda_1) \ll 0$, $L(\lambda_2) \gg 0$ und $\dfrac{\partial L(\lambda)}{\partial \lambda} \gg 0$ für alle $\lambda \in [\lambda_1, \lambda_2]$. Dann gibt es einen Operator $Z \in \mathfrak{R}$ mit den folgenden Eigenschaften:*

1) $L(Z) = 0$;
2) *Z ist ähnlich zu einem selbstadjungierten Operator*;
3) $\sigma(Z) = \sigma(L) \cap [\lambda_1, \lambda_2]$, $\sigma_p(Z) = \sigma_p(L) \cap [\lambda_1, \lambda_2]$, *und die zu einem solchen Eigenwert gehörenden Eigenelemente von Z und L stimmen überein.*

Der Operator Z ist durch die Eigenschaft 1) *und die erste Gleichung von* 3) *eindeutig bestimmt.*

Folgerung. *Ist $\sigma(L) \cap [\lambda_1, \lambda_2]$ eine höchstens abzählbare Menge, so gibt es eine Rieszsche Basis von $\mathfrak{H}$, die aus Eigenelementen von L zu Eigenwerten in $[\lambda_1, \lambda_2]$ besteht; diese Eigenwerte bilden dann zusammen mit ihren Häufungspunkten die Menge $\sigma(L) \cap [\lambda_1, \lambda_2]$, ihre Eigenräume werden von Elementen der Basis aufgespannt.*

Ohne Beschränkung der Allgemeinheit setzen wir im Beweis des Satzes $0 \leqq \lambda_1 \leqq \leqq \lambda_2 < \infty$ voraus. Wir wählen einen Punkt $\alpha \in (\lambda_1, \lambda_2)$ und betrachten neben der Schar L die Scharen L_t:

$$L_t(\lambda) = L(\lambda) + (t-1) L(\alpha), \qquad 0 \leqq t \leqq 1.$$

Offensichtlich gilt $\dfrac{\partial L_t(\lambda)}{\partial \lambda} = \dfrac{\partial L(\lambda)}{\partial \lambda}$, und die Scharen L_t genügen bezüglich $[\lambda_1, \lambda_2]$ denselben Voraussetzungen wie die Schar L.

Im Verlauf des Beweises des Satzes formulieren wir einige Lemmata; dabei seien stets die im Satz über die Schar L getroffenen Voraussetzungen erfüllt.

Lemma 4. *Es gibt eine Umgebung $\mathfrak{A}$ von $[\lambda_1, \lambda_2]$ mit*

$$\mathfrak{A} \setminus [\lambda_1, \lambda_2] \subset \varrho(L_t) \quad \textit{für} \quad 0 \leqq t \leqq 1.$$

Beweis. Offensichtlich gibt es ein $\delta > 0$, so daß für alle $t \in [0, 1]$ folgendes gilt:

$$L_t(\lambda) \ll 0 \quad \text{für} \quad \lambda \in [\lambda_1 - \delta, \lambda_1],$$

$$L_t(\lambda) \gg 0 \quad \text{für} \quad \lambda \in [\lambda_2, \lambda_2 + \delta],$$

$$L'(\lambda) \gg 0 \quad \text{für} \quad \lambda \in [\lambda_1 - \delta, \lambda_2 + \delta].$$

Für $\lambda = \tau + i\varepsilon$, $\tau \in [\lambda_1 - \delta, \lambda_2 + \delta]$, gilt dann

$$L_t(\lambda) = L_t(\tau) + i\varepsilon L'(\tau) + \varepsilon^2 0(1) \qquad (\varepsilon \to 0),$$

wobei das Glied $O(1)$ für $t \in [0, 1]$ gleichmäßig beschränkt ist. Deshalb hat für $|\varepsilon|$ hinreichend klein einer der Operatoren $\pm L_t(\lambda)$ einen streng positiven Imaginärteil, ist also beschränkt invertierbar.

Wir wählen jetzt $\eta > 0$ so, daß der durch die Eckpunkte $\lambda_1 - i\eta \dots \lambda_2 - i\eta \dots \dots \lambda_2 + i\eta \dots \lambda_1 + i\eta \dots \lambda_1 - i\eta$ definierte orientierte Streckenzug $\mathfrak{C}$ ganz in der Umgebung $\mathfrak{A}$ aus Lemma 4 verläuft. Weiter sei

$$K_t^{(j)} = \frac{1}{2\pi i} \int_{\mathfrak{C}} \lambda^j L_t^{-1}(\lambda)\, d\lambda, \quad j = 0, \pm 1, \dots .$$

Auf Grund der Beziehung (5) und Lemma 4 ist $\sigma(L_t) \cap [\lambda_1, \lambda_2]$ eine Spektralmenge für den gemäß (4) gebildeten Operator $\mathbf{L}_t$ und der zugehörige Rieszsche Projektor $\mathbf{E}_t$ gestattet die Matrixdarstellung

$$\mathbf{E}_t = -\frac{1}{2\pi i} \int_{\mathfrak{C}} (\mathbf{L}_t - \lambda \mathbf{I})^{-1}\, d\lambda = \begin{pmatrix} K_t^{(N-1)} & \dots \\ \vdots & \\ K_t^{(1)} & \dots \\ K_t^{(0)} & \dots \end{pmatrix}; \tag{9}$$

wir vermerken noch die später benötige Beziehung

$$(10)\qquad \mathbf{L}_t^j\mathbf{E}_t = -\frac{1}{2\pi i}\int_{\mathfrak{C}} \lambda^j(\mathbf{L}_t-\lambda\mathbf{I})^{-1}\,d\lambda = \begin{pmatrix} K_t^{(N+j-1)} & \dots \\ \vdots & \\ K_t^{(j+1)} & \dots \\ K_t^{(j)} & \dots \end{pmatrix}, \quad j=0, \pm 1, \dots$$

Gemäß der Aussage 1° von Abschnitt 1 ist der Operator $\mathbf{E}_t$ G-selbstadjungiert. Der Wertebereich von $\mathbf{E}_0$ besteht aus allen $\mathbf{x}\in\mathfrak{H}$ der Gestalt

$$\mathbf{x}=\begin{pmatrix}\alpha^{N-1}x\\ \vdots\\ \alpha x\\ x\end{pmatrix}, \quad x\in\mathfrak{H}.$$

Um das zu sehen, beachten wir die für alle λ aus einer hinreichend kleinen Umgebung von $\lambda=\alpha$ bestehende Beziehung

$$L^{-1}(\lambda) = \frac{1}{\lambda-\alpha}L'^{-1}(\alpha)+\dots,$$

wobei die nicht aufgeschriebenen Summanden holomorph von λ abhängen. Aus (6) und (7) folgt dann leicht die Behauptung.

Auf Grund der Beziehung (8) ist der Teilraum $\mathbf{E}_0\mathfrak{H}$ G-positiv. Da $\mathbf{E}_t$ in der gleichmäßigen Operatorentopologie für $0\leqq t\leqq 1$ analytisch von t abhängt, ist gemäß Lemma 1 auch $\mathbf{E}_t\mathfrak{H}$, $0\leqq t\leqq 1$, ein streng G-positiver Teilraum, also ein Hilbertraum für das G-Skalarprodukt. In diesem Hilbertraum ist $\mathbf{L}_t$ G-selbstadjungiert, und es gilt $\sigma(\mathbf{L}_t|\mathbf{E}_t\mathfrak{H})\subset[\lambda_1, \lambda_2]$. Für beliebiges $\mathbf{x}\in\mathfrak{H}$ gilt also

$$[\mathbf{E}_t\mathbf{x}, \mathbf{x}]\geqq 0, \quad \lambda_1[\mathbf{L}_t^{j-1}\mathbf{E}_t\mathbf{x}, \mathbf{x}]\leqq[\mathbf{L}_t^j\mathbf{E}_t\mathbf{x}, \mathbf{x}]\leqq\lambda_2[\mathbf{L}_t^{j-1}\mathbf{E}_t\mathbf{x}, \mathbf{x}].$$

Wenden wir diese Beziehungen auf die Elemente

$$\mathbf{x}=\begin{pmatrix}x\\0\\ \vdots\\0\end{pmatrix}, \quad x\in\mathfrak{H},$$

an, so ergibt sich bei Beachtung von (10)

$$(11)\qquad K_t^{(0)}\geqq 0;\quad \lambda_1 K_t^{(j-1)}\leqq K_t^{(j)}\leqq\lambda_2 K_t^{(j-1)}, \qquad j=0, \pm 1, \pm 2, \dots.$$

Lemma 5. *Es gilt* $K_t^{(0)}\gg 0$ $(0\leqq t\leqq 1)$.

Beweis. Es genügt zu zeigen, daß der Nullpunkt nicht zu $\sigma(K_t^{(0)})$ gehört. Zuerst überlegen wir uns, daß der Nullpunkt kein Eigenwert von $K_t^{(0)}$ ist. Aus $K_t^{(0)}x_0=0$,

$x_0 \neq 0$, folgt gemäß (11) $K_t^{(j)} x_0 = 0$ für $j = 1, 2, \dots$, also ist dann auf Grund von (9) auch $\mathbf{E}_t \mathbf{x}_0 = 0$ für $\mathbf{x}_0 = (x_j)$ mit $x_1 = x_0, x_2 = x_3 = \dots = x_N = 0$. Somit läßt sich $(\mathbf{L}_t - \lambda \mathbf{I})^{-1} \mathbf{x}_0$ auf ganz $\mathfrak{A}$ holomorph fortsetzen, insbesondere gestattet auch $L_t^{-1}(\lambda) x_0$ eine solche Fortsetzung. Wir bezeichnen sie mit $y(\lambda)$. Für $\varphi(\lambda) = (y(\lambda), x_0)$ gilt dann $\varphi(\lambda_1) < 0$, $\varphi(\lambda_2) > 0$, andererseits ist aber für Punkte $\lambda \neq \bar{\lambda}$, $\lambda \in \mathfrak{A}$,

$$\varphi'(\lambda) = -\left(L_t^{-1}(\lambda) L_t'(\lambda) L_t^{-1}(\lambda) x_0, x_0\right) = -\left(L_t'(\lambda) y(\lambda), y(\bar{\lambda})\right),$$

also auch $\varphi'(\lambda) \leqq 0$ für $\lambda \in [\lambda_1, \lambda_2]$.

Um zu zeigen, daß $K_t^{(0)}$ sogar streng positiv ist, führen wir gemäß [11] einen Hilbertraum $\tilde{\mathfrak{H}}$ in folgender Weise ein: Die lineare Menge $\mathfrak{L}$ aller beschränkten Folgen $(x^{(n)})$ von Elementen aus $\mathfrak{H}$ versehen wir mit dem Skalarprodukt

$$\left((x^{(n)}), (y^{(n)})\right) = \operatorname*{LIM}_n \,(x^{(n)}, y^{(n)}),$$

wobei $\operatorname*{LIM}_n$ einen beliebigen, im folgenden festgehaltenen Banachlimes bezeichnet, bilden den Faktorraum nach $\mathfrak{L}_0 = \{(x^{(n)}) \in \mathfrak{L} : ((x^{(n)}), (x^{(n)})) = 0\}$ und vervollständigen diesen. Für einen beschränkten linearen Operator A in $\mathfrak{H}$ bezeichne $\tilde{A}$ den durch die Festsetzung

$$\tilde{A}(x^{(n)}) = (A x^{(n)})$$

in $\tilde{\mathfrak{H}}$ induzierten Operator. Dann gilt für A, $A_j \in \mathfrak{R}$, $j = 1, 2, \dots$: $\widetilde{\alpha_1 A_1 + \alpha_2 A_2} = \alpha_1 \tilde{A}_1 + \alpha_2 \tilde{A}_2$, $\widetilde{A^{-1}} = \tilde{A}^{-1}$ falls $0 \in \varrho(A)$, $\sigma(A) = \sigma_p(\tilde{A})$ falls $A = A^*$, und für eine bezüglich der Operatorennorm gegen A konvergente Folge (A_j) konvergiert auch $(\tilde{A}_j)$ bezüglich der Operatorennorm in $\tilde{\mathfrak{H}}$ gegen $\tilde{A}$.

Die Schar $\tilde{L}_t : \tilde{L}_t(\lambda) = \tilde{L}(\lambda) + (t-1)\tilde{L}(\alpha)$ in $\tilde{\mathfrak{H}}$ genügt denselben Voraussetzungen wie L_t in $\mathfrak{H}$, und es ist gemäß den oben vermerkten Regeln für den Übergang $A \to \tilde{A}$

$$\widetilde{K_t^{(0)}} = \frac{1}{2\pi i} \int_{\mathfrak{C}} \tilde{L}_t^{-1}(\lambda)\, d\lambda.$$

Nach dem ersten Teil des Beweises gilt folglich $0 \notin \sigma_p(\widetilde{K_t^{(0)}})$, d.h. $0 \notin \sigma(K_t^{(0)})$. Damit ist das Lemma bewiesen.

Wir untersuchen den Teilraum $\mathbf{E}_t \mathfrak{H}$ genauer. Zunächst gibt es für hinreichend kleine t Operatoren $\mathbf{T}_t$, die $\mathbf{E}_0 \mathfrak{H}$ eineindeutig und stetig auf $\mathbf{E}_t \mathfrak{H}$ abbilden und zusammen mit ihrer Inversen stetig (bezüglich der gleichmäßigen Operatorentopologie) von t abhängen. Wir setzen

$$\mathfrak{H}_1 = \left\{ \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix} : \quad x \in \mathfrak{H} \right\}$$

und betrachten den Teilraum $\overset{-1}{\mathbf{T}_t}\mathbf{E}_t\mathfrak{H}_1$. Für $t=0$ gilt $\overset{-1}{\mathbf{T}_0}\mathbf{E}_0\mathfrak{H}_1=\mathbf{E}_0\mathfrak{H}$. Dann bildet aber $\overset{-1}{\mathbf{T}_t}\mathbf{E}_t$ auch für hinreichend kleine t den Teilraum $\mathfrak{H}_1$ eineindeutig auf $\mathbf{E}_0\mathfrak{H}$ ab, d.h., es gilt

$$\overset{-1}{\mathbf{T}_t}\mathbf{E}_t\mathfrak{H}_1=\mathbf{E}_0\mathfrak{H} \quad \text{oder} \quad \mathbf{E}_t\mathfrak{H}_1=\mathbf{T}_t\mathbf{E}_0\mathfrak{H}=\mathbf{E}_t\mathfrak{H}.$$

Für die betrachteten t läßt sich $\mathbf{E}_t\mathfrak{H}$ also in der Form

$$\mathbf{E}_t\mathfrak{H}=\left\{\begin{pmatrix} K_t^{(N-1)}x \\ \vdots \\ K_t^{(1)}x \\ K_t^{(0)}x \end{pmatrix} : \quad x\in\mathfrak{H}\right\}$$

schreiben. Auf Grund von Lemma 5 folgt daraus mit $Z_t^{(j)}=K_t^{(j)}(K_t^{(0)})^{-1}$:

$$\mathbf{E}_t\mathfrak{H}=\left\{\begin{pmatrix} Z_t^{(N-1)}x \\ \vdots \\ Z_t^{(1)}x \\ x \end{pmatrix} : \quad x\in\mathfrak{H}\right\}.$$

Anwendung von Lemma 2 ergibt mit $Z_t=Z_t^{(1)}: Z_t^{(j)}=Z_t^j$ und

$$L_t(Z_t)=0 \tag{12}$$

für alle hinreichend kleinen t. Nun hängen sowohl der Operator Z_t als auch die Koeffizienten der Schar L_t für $0\leqq t\leqq 1$ analytisch von t ab. Deshalb besteht die Beziehung (12) auch für $t=1$, d.h., der Operator

$$Z=K_1^{(1)}(K_1^{(0)})^{-1}$$

genügt der Gleichung (2). Offensichtlich ist Z von links symmetrisierbar durch den streng positiven Operator $(K_1^{(0)})^{-1}$. Damit sind die ersten beiden Aussagen des Satzes bewiesen. Die erste Beziehung der Aussage 3) ergibt sich aus den Gleichungen (beachte Lemma 2)

$$\sigma(Z)=\sigma(\mathbf{L}|\mathbf{E}_1\mathfrak{H})=\sigma(\mathbf{L})\cap[\lambda_1,\lambda_2]=\sigma(L)\cap[\lambda_1,\lambda_2];$$

die Beweise der übrigen Aussagen von 3) überlassen wir dem Leser.

Genügt Z' der Gleichung $L(Z')=0$ und gilt $\sigma(Z')\subset[\lambda_1,\lambda_2]$, so ist der Teilraum $\mathfrak{E}_{Z'}$ invariant unter $\mathbf{L}$ und es gilt $\sigma(\mathbf{L}|\mathfrak{E}_{Z'})\subset[\lambda_1,\lambda_2]$. Daraus folgt $\mathfrak{E}_{Z'}\subset\mathfrak{E}_Z$, woraus sich leicht $Z=Z'$ ergibt. Damit ist Satz 1 bewiesen.

5. Aus Satz 1 folgern wir jetzt ohne Schwierigkeit den

Satz 2. *Gegeben sei die Schar*

$$M(\mu)=A-\mu I-\mu^2 B(\mu), \quad B(\mu)=\sum_{j=2}^{N}\mu^{j-2}B_j \tag{13}$$

mit einem Operator $A \in \mathfrak{R}$, $A \geqq 0$ und einem Operatorpolynom $B(\mu)$, das für $\mu \geqq 0$ zusammen mit seiner Ableitung $B'(\mu)$ nichtnegative Werte (in $\mathfrak{R}$) annimmt. Dann gibt es einen Operator $Y \in \mathfrak{R}$ mit den folgenden Eigenschaften:

1) $M(Y)=0$;
2) *Y ist ähnlich zu einem selbstadjungierten Operator;*
3) $\sigma(Y) = \sigma(M) \cap [0, \infty)$, $\sigma_p(Y) = \sigma_p(M) \cap [0, \infty)$, *und die zu einem solchen Eigenwert gehörenden Eigenelemente von Y und M stimmen überein.*

Folgerung. *Ist $\sigma(M) \cap [0, \infty)$ eine höchstens abzählbare Menge, so gibt es eine Rieszsche Basis von $\mathfrak{H}$, die aus Eigenelementen von M zu Eigenwerten in $[0, \infty)$ besteht; diese Eigenwerte bilden zusammen mit ihren Häufungspunkten die Menge $\sigma(M) \cap [0, \infty)$, ihre Eigenräume werden von Elementen dieser Basis aufgespannt.*

Beweis. Aus der Gestalt der Schar M ersieht man leicht, daß Zahlen μ_1, μ_2 und η mit $\mu_1 < 0$; $\|A\| < \mu_2 < \eta$ so gewählt werden können, daß

$$M(\mu_1) \gg 0, \quad M(\mu_2) \ll 0$$

und

$$N \frac{\eta}{\eta - \mu} M(\mu) + \eta M'(\mu) \ll 0 \quad \text{für} \quad \mu_1 \leqq \mu \leqq \mu_2 \tag{14}$$

gilt. Wir betrachten die Schar

$$L(\lambda) = -(1+\lambda)^N F^{-1/2} M\left(\frac{\eta\lambda}{1+\lambda}\right) F^{-1/2} \quad \text{mit} \quad F = -M(\eta).$$

Dann gilt für $\lambda_i = \dfrac{\mu_i}{\eta - \mu_i}$, $i = 1, 2$,

$$L(\lambda_1) = -(1+\lambda_1)^N F^{-1/2} M(\mu_1) F^{-1/2} \ll 0;$$

$$L(\lambda_2) = -(1+\lambda_2)^N F^{-1/2} M(\mu_2) F^{-1/2} \gg 0;$$

$$L'(\lambda) = -(1+\lambda)^{N-2} F^{-1/2} \left[N(1+\lambda) M\left(\frac{\eta\lambda}{1+\lambda}\right) + \eta M'\left(\frac{\eta\lambda}{1+\lambda}\right)\right] F^{-1/2}.$$

Durchläuft λ das Intervall $[\lambda_1, \lambda_2]$, so ist $1+\lambda = \dfrac{\eta}{\eta - \mu}$ stets positiv, und für den Ausdruck in der eckigen Klammer auf der rechten Seite ergibt sich mit $\mu = \dfrac{\eta\lambda}{1+\lambda}$:

$$N \frac{\eta}{\eta - \mu} M(\mu) + \eta M'(\mu), \quad \mu_1 \leqq \mu \leqq \mu_2,$$

also ist $L'(\lambda)$ auf $[\lambda_1, \lambda_2]$ wegen (14) streng positiv. Schließlich ist der Koeffizient von λ^N die identische Abbildung.

Die Schar L genügt also allen Voraussetzungen von Satz 1. Daraus folgt die Existenz eines Operators Z mit $L(Z)=0$ und $\sigma(Z)\subset[\lambda_1, \lambda_2]$. Wegen $-1<\lambda_1$ existiert der Operator $(I+Z)^{-1}\in\mathfrak{R}$, und man überzeugt sich leicht davon, daß

$$Y=\eta F^{-1/2}Z(I+Z)^{-1}F^{1/2}$$

die im Satz genannten Eigenschaften hat.

Bemerkung. *Hat A aus* (13) *die Gestalt $A=aI+A_1$ mit $a\geqq 0$ und einem vollstetigen Operator A_1, so ist das Spektrum von Y im Intervall (a, ∞) diskret; insbesondere zieht die Vollstetigkeit von A stets die Vollstetigkeit von Y nach sich.*

Zum Beweis dieser Aussage genügt es zu zeigen, daß für $\lambda>a$ und jede beschränkte Folge (x_n) mit $(Y-\lambda I)x_n \to 0$ (stark), $x_n\to 0$ (schwach) für $n\to\infty$ auch $\|x_n\|\to 0$ folgt. Aus den Voraussetzungen ergibt sich aber

$$(a-\lambda)\|x_n\|^2-\lambda^2(B(\lambda)x_n, x_n)\to 0,$$

woraus leicht die Behauptung folgt.

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Strictly cyclic shifts on l_p

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1. Introduction. Let X be a complex Banach space and let $\mathcal{A}$ be a closed abelian subalgebra of $\mathcal{B}(X)$, the algebra of all bounded linear transformations on X. $\mathcal{A}$ is said to be strictly cyclic if there is a vector x_0 in X such that $\mathcal{A}x_0 = X$. General properties and examples of strictly cyclic algebras may be found in [1] and [4]. A large class of examples is given by the algebras generated by certain weighted shifts. In this paper we will be concerned with characterizing strictly cyclic weighted shifts on l_p. (An operator T is said to be strictly cyclic if the closed subalgebra it generates is strictly cyclic.)

For $1 \leqq p < \infty$ let l_p be the Banach space of all absolutely p-summable sequences of complex numbers. Let $\{e_0, e_1, \dots\}$ be the standard basis for l_p. For each bounded sequence $\alpha = \{\alpha_1, \alpha_2, \dots\}$ of non-zero complex numbers the operator S_α in $\mathcal{B}(l_p)$ defined by $S_\alpha\left(\sum_{n=0}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} \alpha_n x_{n-1} e_n$ is called the weighted shift on l_p with weight sequence α. It is well known that $\|S_\alpha\| = \sup_n |\alpha_n|$. We set $\beta_0 = 1$ and $\beta_n = \alpha_1 \alpha_2 \dots \alpha_n$ for $n \geqq 1$. In [1] MARY EMBRY showed that S_α is strictly cyclic on l_1 if and only if $\sup_{n,m} \left|\frac{\beta_{n+m}}{\beta_n \beta_m}\right| < \infty$. Although this is not valid for $p > 1$, we determine in this paper a dual result for shifts on l_p.

In § 2 we establish the basic notation and concepts used throughout the paper. Section 3 is concerned with strictly cyclic shifts on l_p for $1 < p < \infty$. We give a general sufficient condition for strict cyclicity. Then shifts on l_p whose weights are monotone non-increasing in modulus are completely characterized. Several tests for strict cyclicity are given as corollaries to these results.

In § 4 weighted shifts on l_∞ are examined. The commutants of such shifts are characterized, with special emphasis on those shifts S_α such that $\inf_n |\alpha_n| > 0$. To each weighted shift S_α on l_∞ there is associated in a natural way a closed abelian subalgebra $\mathcal{B}$ of the commutant of S_α. We obtain a necessary and sufficient condition for strict cyclicity for $\mathcal{B}$, analogous to our results for the case $p < \infty$. In conclusion we list a number of open questions relating to this material.

2. Preliminaries. The following facts about strictly cyclic abelian algebras will be used throughout this paper. Details can be found in [1]. Let x_0 be a strictly cyclic vector for the closed abelian subalgebra $\mathscr{A}$ of $\mathscr{B}(X)$. Then for each x in X there is a unique operator A_x in $\mathscr{A}$ such that $A_x x_0 = x$. The map $x \to A_x$ is a linear homeomorphism of X onto $\mathscr{A}$. Therefore there is a constant M such that for all x and y in X, $\|A_x y\| \leqq M\|x\|\,\|y\|$. We will concern ourselves with $X = l_p$.

For A in $\mathscr{B}(l_p)$ we let $\mathscr{A}(A)$ be the weakly closed subalgebra of $\mathscr{B}(l_p)$ generated by A and the identity operator I. That is, $\mathscr{A}(A)$ is the weak closure of the set of polynomials in A. We then let $\mathscr{A}'(A) = \{B \text{ in } \mathscr{B}(l_p): AB = BA\}$, called the commutant of A. It is well known that for every shift S_α on l_p, $\mathscr{A}'(S_\alpha)$ is a maximal abelian subalgebra of $\mathscr{B}(l_p)$ and e_0 is a cyclic vector for $\mathscr{A}(S_\alpha)$. Thus it follows from [3; Cor. 3. 3] that S_α is strictly cyclic if and only if e_0 is strictly cyclic for S_α. It is easy to see that any operator similar to a strictly cyclic operator is itself strictly cyclic, and that an argument completely analogous to [2; Th. 1] shows that S_α is similar, via an isometric isomorphism, to S_γ where $\gamma_n = |\alpha_n|$, $n = 1, 2, \dots$. Therefore, when convenient, we will assume our shifts to have positive weights.

Lemma 2. 1. *S_α is strictly cyclic on l_p if and only if*

$$\sum_{n=0}^{\infty} \left| \sum_{m=0}^{n} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right|^p < \infty \tag{1}$$

for all x and y in l_p.

Proof. Suppose S_α is strictly cyclic on l_p. Let y be in l_p and for each positive integer N let $A_N = \sum_{n=0}^{N} \frac{y_n}{\beta_n} S_\alpha^n$. Then A_N is in $\mathscr{A}(S_\alpha)$ and $\|(A_y - A_N)e_0\| = \left\| y - \sum_{n=0}^{N} y_n e_n \right\|$. Thus A_N converges in norm to A_y, and so $A_y = \sum_{n=0}^{\infty} \frac{y_n}{\beta_n} S_\alpha^n$, the series converging in the operator norm. Now, for each x and y in l_p,

$$A_y x = \sum_{n=0}^{\infty} \frac{y_n}{\beta_n} S_\alpha^n x = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_m y_n \frac{\beta_{n+m}}{\beta_n \beta_m} e_{n+m} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{\beta_n}{\beta_n \beta_{n-m}} x_m y_{n-m} \right) e_n .$$

Therefore, (1) holds.

Conversely, suppose (1) holds for each x and y in l_p. For each x in l_p let T_x be the linear transformation on l_p given by

$$T_x y = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \right) e_n .$$

T_x is easily seen to be a closed linear transformation and so by the closed graph theorem T_x is bounded. Moreover, for each x and y in l_p, $T_x y = T_y x$. Thus $\|T_y x\| \leqq \leqq \|T_x\|\,\|y\|$ and by the uniform boundedness principle there exists a constant M such that $\|T_y x\| \leqq M\|x\|\,\|y\|$ for all x and y in l_p. For each non-negative integer N let

$y^{(N)}=\sum_{n=0}^{N} y_n e_n$. Then $\lim_{N\to\infty} \|T_y - T_{y^{(N)}}\| = 0$. But $T_{y^{(N)}} = \sum_{n=0}^{N} \frac{y_n}{\beta_n} S_\alpha^n$ and so T_y is a member of $\mathscr{A}(S_\alpha)$. Since $T_y e_0 = y$, S_α is strictly cyclic.

3. The case $1<p<\infty$. The following lemma gives the most general sufficient condition for strict cyclicity known to the authors at this time. This condition has been discovered independently by Mary Embry. Throughout this section we assume $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$.

Lemma 3. 1. (Nikol'skiĭ [5]) *Let S_α be a weighted shift on l_p and suppose $M=\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q<\infty$. Then S_α is strictly cyclic on l_p.*

Proof. Let x and y be in l_p. Then by Hölder's inequality,

$$\sum_{n=0}^{\infty}\left|\sum_{m=0}^{n}\frac{\beta_n}{\beta_m\beta_{n-m}}x_m y_{n-m}\right|^p \leqq \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q\right)^{p/q}\left(\sum_{m=0}^{n}|x_m|^p|y_{n-m}|^p\right)\leqq$$

$$\leqq M^{p/q}\sum_{n=0}^{\infty}\sum_{m=0}^{n}|x_m|^p|y_{n-m}|^p = M^{p/q}\|x\|^p\|y\|^p.$$

By Lemma 2. 1, S_α is strictly cyclic.

We show now that under the assumption of monotonicity of the weights the converse to Lemma 3. 1 is valid.

Theorem 3. 2. *If $\{|\alpha_n|\}$ is monotonically non-increasing then S_α is strictly cyclic on l_p if and only if $\sup_n \sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q<\infty$.*

Proof. Suppose S_α is strictly cyclic on l_p and $\{|\alpha_n|\}$ is monotonically non-increasing. By the remarks in section 2 we assume without loss of generality that each $\alpha_n>0$. By Lemma 2. 1 $\exists M>0$ such that $\sum_{n=0}^{\infty}\left|\sum_{m=0}^{n}\frac{\beta_n}{\beta_m\beta_{n-m}}x_m y_{n-m}\right|^p \leqq M\|x\|^p\|y\|^p$ for all x and y in l_p. Let x and y be in l_p with $x_n\geqq 0$ and $y_n\geqq 0$ for each n. For each positive integer N,

$$\sum_{n=N}^{2N}\left[\sum_{m=0}^{N}\frac{\beta_n}{\beta_m\beta_{n-m}}x_m y_{n-m}\right]^p\leqq M\|x\|^p\|y\|^p.$$

Since $\{\alpha_n\}$ is monotonically decreasing, $\frac{\beta_n}{\beta_{n-k}}\leqq\frac{\beta_m}{\beta_{m-k}}$ whenever $0\leqq k\leqq m\leqq n$. Therefore, replacing $\frac{\beta_n}{\beta_{n-m}}$ by $\frac{\beta_{2N}}{\beta_{2N-m}}$ in the above inequality we see that $\sum_{n=N}^{2N}\left[\sum_{m=0}^{N}\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}x_m y_{n-m}\right]^p\leqq M\|x\|^p\|y\|^p$. Let $y_k=\left(\frac{1}{2N+1}\right)^{1/p}$ for $0\leqq k\leqq 2N$ and $y_k=0$ otherwise. Then the preceding inequality reduces to

$$\frac{N+1}{2N+1}\left[\sum_{m=0}^{N}\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}x_m\right]^p \leqq M\|x\|^p.$$

Hence for every x in l_p, $\left|\sum_{m=0}^{N}\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}x_m\right| \leqq (2M)^{1/p}\|x\|$. It follows that

$$\sum_{m=0}^{N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q \leqq (2M)^{q/p} \equiv C.$$

Now,

$$\sum_{m=0}^{2N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q < \sum_{m=0}^{N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q + \sum_{m=N}^{2N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q = 2\sum_{m=0}^{N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q \leqq 2C.$$

On the other hand

$$\sum_{m=0}^{2N+1}\left(\frac{\beta_{2N+1}}{\beta_m\beta_{2N+1-m}}\right)^q = 1+\sum_{m=0}^{2N}\left(\frac{\beta_{2N+1}}{\beta_m\beta_{2N+1-m}}\right)^q =$$

$$= 1+\sum_{m=0}^{2N}\left(\frac{\alpha_{2N+1}}{\alpha_{2N+1-m}}\right)^q\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q \leqq 1+\sum_{m=0}^{2N}\left(\frac{\beta_{2N}}{\beta_m\beta_{2N-m}}\right)^q \leqq 1+2C.$$

Thus we see that $\sup_n \sum_{m=0}^{n}\left(\frac{\beta_n}{\beta_m\beta_{n-m}}\right)^q \leqq 1+2C<\infty$, completing the proof.

Remark. The argument above is valid under the somewhat weaker assumption that $\{|\alpha_n|\}$ is ultimately monotone non-increasing.

Lemma 3. 1 and Theorem 3. 2 admit the following interesting corollaries. The first of these generalizes [4; Th. 4. 1].

Corollary 3. 3. *Suppose there exist u and v in l_q such that for all n and m, $\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right| \leqq |u_n|+|v_m|$. Then S_α is strictly cyclic on l_p.*

Proof. For $n\geqq m\geqq 0$, $\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q \leqq 2^q(|u_m|^q+|v_{n-m}|^q)$ and hence

$$\sup_n \sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|^q \leqq 2^q(\|u\|_q^q+\|v\|_q^q)<\infty.$$

Corollary 3.4. *Suppose $\{|\alpha_n|\}$ is monotonically non-increasing and $\sum_{m=0}^{\infty}\left|\frac{\beta_{2m}}{\beta_m^2}\right|^q<\infty$. Then S_α is strictly cyclic on l_p.*

Proof. It is an easy consequence of the monotonicity assumption that for $i\geqq j\geqq 0$, $\left|\frac{\beta_{i+j}}{\beta_i\beta_j}\right| \leqq \left|\frac{\beta_{2j}}{\beta_j^2}\right|$ and so for any i and $j\geqq 0$, $\left|\frac{\beta_{i+j}}{\beta_i\beta_j}\right| \leqq \left|\frac{\beta_{2i}}{\beta_i^2}\right|+\left|\frac{\beta_{2j}}{\beta_j^2}\right|$. By Corollary 3. 3, S_α is strictly cyclic.

The next corollary shows that the collection of strictly cyclic shifts on l_p is fairly large.

Corollary 3. 5. *Suppose $\{|\alpha_n|\}$ is monotonically non-increasing and for some $r>0$, $\sum_{n=1}^{\infty} |\alpha_n|^r<\infty$. Then S_α is strictly cyclic on l_p for every $p \geqq 1$.*

Proof. Fix $p>1$ (the case $p=1$ follows from EMBRY's result mentioned above). We show that the hypothesis of Corollary 3. 3 is satisfied. Let M be a positive integer such that $Mq \geqq r$. Then if n and m are non-negative integers with $m \geqq M$ we have

$$\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right| = \frac{1}{|\beta_M|}\,|\alpha_{n+1}\cdots\alpha_{n+M}|\left|\frac{\alpha_{n+M+1}\cdots\alpha_{n+m}}{\alpha_{M+1}\cdots\alpha_m}\right|.$$

It follows from the assumption of monotonicity that

$$\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right| \leqq \frac{1}{|\beta_M|}\,|\alpha_{n+1}|^M.$$

Let $u_k = \max\limits_{n,m<M}\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right|$ if $0 \leqq k<M$ and $u_k = \frac{1}{|\beta_M|}\,|\alpha_{k+1}|^M$ otherwise. Then

$$\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right| \leqq u_n+u_m \text{ for all } n \text{ and } m \geqq 0.$$

Moreover since $Mq>r$, $\Sigma|\alpha_n|^{Mq}<\infty$ and consequently $\{u_n\}$ is in l_q. By Corollary 3. 3, S_α is strictly cyclic.

Only slight modifications of [3; Cor. 4. 8] show that if $\alpha_n = \dfrac{1}{\log(n+1)}$ for each $n \geqq 1$ then S_α is strictly cyclic on l_p for all $p>1$. However $\{\alpha_n\}$ decreases monotonically to 0 and is not r summable for any $r>0$.

We point out now a common theme in Embry's result concerning shifts on l_1 and our results for $p>1$. For $n \geqq 0$ let e_n' be the sequence $(0, 0, \ldots, 1, 0, 0, \ldots)$, the 1 in the n^{th} position (beginning the indexing at 0). For $p>1$ we consider $\{e_n'\}$ as the standard basis for l_q. For $p=1$ we may still write every element of l_∞ uniquely in the form $\sum_{n=0}^{\infty} a_n e_n'$. Now for $N \geqq 0$ let $f_N = \sum_{n=0}^{N} \frac{\beta_N}{\beta_n\beta_{N-n}}\, e_n'$, viewed as a continuous linear functional on l_p. Then for $p=1$, $\|f_N\|_\infty = \max\limits_{0\leqq n\leqq N}\left|\frac{\beta_N}{\beta_n\beta_{N-n}}\right|$ and so Mary Embry's result can be rephrased in the following manner.

Theorem. (EMBRY) *S_α is strictly cyclic on l_1 if and only if $\{f_N\}$ is bounded.*

Our result above reduces to:

If $p>1$ and $\{f_N\}$ is bounded then S_α is strictly cyclic on l_p. The converse holds if $\{|\alpha_n|\}$ is monotonically non-increasing.

If $1<p<\infty$ and $\{|\alpha_n|\}$ is monotonically non-increasing then essentially the same proof as [6; Cor. 1] shows that the spectral radius r of S_α is $\lim_{N\to\infty} |\alpha_n|$. Now suppose $\{|\alpha_n|\}$ is monotonically non-increasing and S_α is strictly cyclic on l_p. Then $\{f_N\}$ is bounded in l_q. For fixed m and $N \geqq m$,

$$\langle f_N, e'_m\rangle = \frac{\beta_N}{\beta_m \beta_{N-m}} = \frac{1}{\beta_m}(\alpha_{N-m+1}\cdots\alpha_N),$$

hence for each m, $\lim_{N\to\infty} \langle f_N, e'_m\rangle = \dfrac{r^m}{\beta_m}$. It follows that f_N converges weakly to $\sum_{m=0}^{\infty} \dfrac{r^m}{\beta_m} e'_m$ in l_q.

We will see in the next section how some of these ideas may be extended to l_∞.

4. The case $p=\infty$. Since l_∞ is not separable there are no cyclic operators on l_∞. However for a bounded sequence $\{\alpha_n\}$ of complex numbers S_α still defines a bounded operator on l_∞ and we may ask when $\mathscr{A}'(S_\alpha)$ is strictly cyclic. First note that if T is a linear transformation from l_∞ to l_∞ then there is a sequence $\{t_0, t_1, \ldots\}$ of continuous linear functionals on l_∞ such that for every x in l_∞, $Tx = \langle t_0(x), t_1(x), \ldots\rangle$. Moreover, T is bounded if and only if $\sup_n \|t_n\| < \infty$. If this holds then $\|T\| = \sup_n \|t_n\|$. An easy computation shows that if T is a bounded linear operator on l_∞ with $\{t_n\}$ defined as above, then $TS_\alpha = S_\alpha T$ if and only if

(2) $$t_0 \circ S_\alpha = 0 \quad \text{and} \quad t_{n+1} \circ S_\alpha = \alpha_{n+1} t_n \qquad (n=0, 1, \ldots).$$

We now examine a special class of operators in $\mathscr{A}'(S_\alpha)$. Let

$$\mathscr{E} = \left\{x \text{ in } l_\infty\colon \sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| |x_m| < \infty\right\}.$$

For each x in $\mathscr{E}$ define the linear transformation A_x on l_∞ by

$$(A_x y)_n = \sum_{m=0}^{n} \frac{\beta_n}{\beta_m \beta_{n-m}} x_m y_{n-m} \qquad (n=0, 1, 2, \cdots).$$

With $e_0=(1, 0, 0, \ldots)$, etc., it is easily seen that $A_x e_0 = x$, $A_{e_1} = \dfrac{1}{\alpha_1} S_\alpha$, $A_{e_0}=1$, A_x is bounded, and

$$\|A_x\| = \sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| |x_m|.$$

Let $\mathscr{B} = \{A_x\colon x \text{ in } \mathscr{E}\}$.

Lemma 4.1. *Let x and y be in $\mathscr{E}$. Then $z = A_x y$ is in $\mathscr{E}$ and $A_z = A_x A_y = A_y A_x$.*

Proof. We must show that $\sup_n \sum_{m=0}^{n} \left|\dfrac{\beta_n}{\beta_m \beta_{n-m}} z_m\right| < \infty$, where

$$z_k = \sum_{l=0}^{k} \frac{\beta_k}{\beta_l \beta_{k-l}} x_l y_{k-l}.$$

Fix an integer $n \geqq 0$. Then

$$\sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\sum_{k=0}^{m}\frac{\beta_m}{\beta_k\beta_{n-k}}x_k y_{m-k}\right| \leqq \sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|\sum_{k=0}^{m}\left|\frac{\beta_m}{\beta_k\beta_{m-k}}\right||x_k||y_{m-k}| =$$

$$=\sum_{k=0}^{n}\left(\sum_{m=k}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right|\left|\frac{\beta_m}{\beta_k\beta_{m-k}}\right||y_{m-k}|\right)|x_k| = \sum_{k=0}^{n}\left|\frac{\beta_n}{\beta_k\beta_{n-k}}\right|\left(\sum_{m=k}^{n}\left|\frac{\beta_{n-k}}{\beta_{n-m}\beta_{m-k}}\right||y_{m-k}|\right)|x_k| =$$

$$= \sum_{k=0}^{n}\left|\frac{\beta_n}{\beta_k\beta_{n-k}}\right|\left(\sum_{m=0}^{n-k}\left|\frac{\beta_{n-k}}{\beta_{n-k-m}\beta_m}\right||y_m|\right)|x_k| \leqq \|A_x\|\,\|A_y\|.$$

Techniques of rearrangement of series similar to those used above show that for each w in l_∞ and each non-negative integer n, $(A_z w)_n=(A_x A_y w)_n=A_y A_x w)_n$, i.e. $A_z=A_x A_y=A_y A_x$. Since $A_x+A_y = A_{x+y}$ we have proved part of the following result.

Theorem 4. 2. *$\mathscr{B}$ is a norm closed abelian subalgebra of $\mathscr{A}'(S_\alpha)$ with S_α and I in $\mathscr{B}$. Moreover, if $\inf_n |\alpha_n|>0$, then $\mathscr{A}'(S_\alpha)=\mathscr{B}$.*

Proof. Let $\{x^{(N)}\}$ be a sequence in $\mathscr{E}$ and let A be a bounded operator on l_∞ such that $\lim_N \|A_{x^{(N)}}-A\| = 0$. Then $x^{(N)}=A_{x^{(N)}}e_0 \to x=Ae_0$. Choose $M>0$ such that $\|A_{x^{(N)}}\|\leqq M$ for all N, i.e. $\sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right| |x_m^{(N)}|\leqq M$ for all N and $n\geqq 0$. Letting $N\to\infty$ we see that $\sup_n \sum_{m=0}^{n}\left|\frac{\beta_n}{\beta_m\beta_{n-m}}\right| |x_m|\leqq M$ hence x is in $\mathscr{E}$ and $A_x=A$.

Now suppose $\delta = \inf_n |\alpha_n|>0$. It is then immediate that the range of S_α^k is $\{x \text{ in } l_\infty : x_i=0 \text{ for } 0\leqq i\leqq k\}$. Let T be in $\mathscr{A}'(S_\alpha)$ with $\{t_n\}$ as in (2). Set $u=Te_0$. We show that u is in $\mathscr{E}$ and $A_u=T$. Let x be in l_∞ and write $x=x_0e_0+S_\alpha z$ for some z in l_∞. Then $t_0(x)=x_0t_0(e_0)=x_0u_0$. Now for $n\geqq 1$

$$t_n\circ S_\alpha^n=\alpha_n t_{n-1}\circ S_\alpha^{n-1}=\cdots=\beta_n t_0$$

so $t_n\circ S_\alpha^{n+1}=0$ for all n. Then $t_n(x)=t_n\left(\sum_{m=0}^{n} x_m e_m\right)=\sum_{m=0}^{n} x_m t_n(e_m)$. Now for $m\geqq 1$,

$$t_n(e_m) = t_n\left(\frac{1}{\alpha_m}S_\alpha e_{m-1}\right) = \frac{1}{\alpha_m}t_n(S_\alpha e_{m-1}) = \frac{\alpha_n}{\alpha_m}t_{n-1}(e_{m-1}) = \cdots$$

$$\cdots = \frac{\alpha_n\cdots\alpha_{n-m+1}}{\alpha_m\cdots\alpha_1}t_{n-m}(e_0) = \frac{\beta_n}{\beta_m\beta_{n-m}}u_{n-m}.$$

Therefore $t_n(x)=\sum_{m=0}^{n}\frac{\beta_n}{\beta_m\beta_{n-m}}x_m u_{n-m}$. Thus u is in $\mathscr{E}$ and $A_u=T$, completing the proof.

Unlike the case $p<\infty$ the vector e_0 need not be cyclic for $\mathscr{A}'(S_\alpha)$. For example if $\alpha_n=1$ for each n then by Theorem 4. 2, $\mathscr{A}'(S_\alpha)=\mathscr{B}$, and so $\mathscr{A}'(S_\alpha)e_0=\mathscr{E}$. But in this case

$$\mathscr{E}=\left\{x \text{ in } l_\infty:\ \sum_{m=0}^{n}|x_m|<\infty\right\}=l_1$$

and hence is not dense in l_∞. However there are many shifts on l_∞ for which e_0 is in fact strictly cyclic for $\mathscr{B}$, and these can be classified precisely, in a manner analogous to Lemma 3. 1.

Proposition 4. 3. *The vector e_0 is strictly cyclic for $\mathscr{B}$ if and only if* $\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| < \infty$.

Proof. Suppose $M = \sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| < \infty$. Let x be in l_∞. Then

$$\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| |x_m| \leqq M \|x\|$$

and hence x is in $\mathscr{E}$, so that $\mathscr{E} = l_\infty$. Conversely, suppose $\mathscr{E} = l_\infty$. Then in particular for $x = (1, 1, 1, \cdots)$ in $\mathscr{E}$, we have $\sup_n \sum_{m=0}^{n} \left|\frac{\beta_n}{\beta_m \beta_{n-m}}\right| < \infty$.

Some open questions:

1. Is the converse to Lemma 3. 1 valid?
2. If $\lim \alpha_n = 0$ need S_α be strictly cyclic on l_p, $1 < p < \infty$?
3. If S_α is a weighted shift on l_∞ is $\mathscr{A}'(S_\alpha)$ abelian?

Added in Proof. A negative answer to question 1 has recently been obtained by G. Fricke. Using Theorem 3. 2, R. Gellar and E. Azoff independently provided negative answers to question 2.

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On p-pure subgroups of torsion-free cotorsion groups

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1. Introduction. In this paper we give an explicit form for the structure of torsion-free cotorsion groups (Theorem 1). We apply this to a special class of groups, the torsion-free abelian groups without elements of infinite p-height. A torsion-free abelian group G has an element a of infinite p-height if the equation $p^n x = a$ is solvable in G for any integer $n \geqq 1$ (p a prime). BOYER and MADER [5] have determined the structure of a torsion-free abelian group G without elements of infinite p-height in terms of p-pure and p-basic subgroups. With the aid of the torsion-free cotorsion groups we state the torsion-free part of their result more precisely (Theorem 2). Then we investigate the p-pure subgroups of groups G without elements of infinite p-height which have the additional property that G is complete with respect to the p-adic topology, the so-called p-closed groups. The similarity with the closed p-groups defined by FUCHS for the torsion case is obvious ([6], p. 114) and one can easily prove the analogues of theorems of p-groups for the torsion-free case (Lemma 3 and 4). Our main object is, however, to derive results on the extensions of homomorphisms for p-pure subgroups of torsion-free cotorsion groups. Our theorems 3 and 4 are generalizations of corresponding results of ARMSTRONG [1] for p-pure subgroups of the group of p-adic integers. Let S be a p-pure subgroup of a p-closed group G and let B be a p-basic subgroup of S. Then $\mathrm{Hom}\,(S/B, G/S)=0$ (or equivalently $\mathrm{Ext}\,(S/B, S)=0$) is a sufficient condition that every $\alpha \in \mathrm{Hom}\,(B, S)$ has an extension to an endomorphism of S. Therefore we investigate the groups S with $\mathrm{Hom}\,(S/B, G/S)=0$. In Theorems 5 and 6 we give some equivalent statements for the condition $\mathrm{Hom}\,(S/B, G/S)=0$. It turns out that, if the rank of $S \leqq \aleph_0$, S is completely decomposable into a direct sum of copies of some torsion-free quotient-divisible group of rank 1. Finally we investigate the groups S as above but without the restriction $\mathrm{Hom}\,(S/B, G/S)=0$. In theorem 7 the structure of these groups is reduced to the case where S is a subgroup of $Z(p)$ containing 1 and with the property that S, as a ring, is a subring of $Z(p)$.

Our notation and terminology is, for the main part, in accordance with that of FUCHS [6]; for unexplained notions we refer to his book [6].

2. The structure of torsion-free cotorsion groups. Torsion-free cotorsion groups were defined by HARRISON [9]. For convenience let us summarize some results of [9]. The word group will always mean abelian group. The additive group of rationals is denoted by Q, the additive group of integers by Z. A group G is reduced if it has no non-trivial divisible subgroup. A reduced group G is called *cotorsion* if G a subgroup of a group M with M/G torsion-free imply that G is a direct summand of M, i.e. $\operatorname{Ext}(H, G)=0$ for all torsion-free groups H.

(i) There is a one-to-one correspondence between all divisible torsion groups and all torsion-free cotorsion groups. If D is a divisible torsion group, the correspondence is $D \to \operatorname{Hom}(Q/Z, D)$. If G is torsion-free cotorsion, the inverse of this correspondence is $G \to (Q/Z) \otimes G$.

A result of FUCHS [7, p. 123] states:

(ii) A torsion-free group is a cotorsion group if and only if it is a reduced algebraically compact group.

In [9, Prop. 2. 1., p. 371] it is proved:

(iii) A group is torsion-free cotorsion if and only if it is isomorphic to a direct summand of a complete (unrestricted) direct sum of p-adic integers.

Definition. The *height* of the p-adic integer π is the integer $k(\geqq 0)$ such that $\pi \in p^k Z(p)$, but $\pi \notin p^{k+1} Z(p)$, where $Z(p)$ is the group of p-adic integers. In order to find the structure of torsion-free cotorsion groups it is enough to determine the groups $\operatorname{Hom}(Q/Z, D)$ for arbitrary divisible torsion groups D (by (i)). The following theorem holds:

Theorem 1. *Let D be a divisible torsion group and suppose $D=\sum_{p_i}\sum_{\alpha_{p_i}} C(p_i^\infty)$, where $C(p_i^\infty)$ is the quasi-cyclic group of type p_i (p_i a prime). Then*

$$\operatorname{Hom}(Q/Z, D) \cong \sum_{p_j}{}^{*} \sum_{\alpha_{p_j}}{}' Z(p_j) \tag{1}$$

where the first (complete) sum Σ^* is taken over all prime numbers p_j and, for each prime number p_j, the number of components π_λ with height$=k$ in $\langle \dots, \pi_\lambda, \dots \rangle \in \sum_{\alpha_{p_j}} Z(p_j)$ is finite ($k=0, 1, 2, \dots$).

A proof of Theorem 1 is given in [11]. All torsion-free cotorsion groups have the structure (1) of an interdirect sum of groups of p-adic integers for different primes p and by the result of FUCHS [7, p. 123, (j)] the torsion-free reduced algebraically compact groups have this form. The following remarks are due to Prof. L. FUCHS.

We have $\Sigma Z(p) \subset \Sigma' Z(p) \subset \Sigma^* Z(p)$, where Σ'/Σ is the maximal divisible subgroup of Σ^*/Σ, the latter group being again algebraically compact. Actually, Σ' is the completion of Σ is the n-adic topology (cf. [9], p. 379), so Σ is dense is Σ' which means Σ'/Σ is divisible ([9], p. 380). Thus Σ^* will be the direct sum of Σ' and a reduced algebraically compact group $\cong \Sigma^*/\Sigma'$. Now we are going to use the concept of p-basic subgroup (p a prime), introduced in [7] by Fuchs. Let G be an arbitrary torsion-free abelian group. Then B_0 is called a *p-basic subgroup* of G, if the following conditions are satisfied:

(i) B_0 is a direct sum of infinite cyclic groups.

(ii) B_0 is a *p-pure subgroup* of G, i.e. $p^r B_0 = B_0 \cap p^r G$ for $r=0, 1, 2, \dots$.

(iii) The factor group G/B_0 is *p-divisible:* i.e. $p^n \bar{x} = \bar{a}$ is solvable in G/B_0 for any $\bar{a} \varepsilon G/B_0$ and any integer $n \geqq 0$.

In [7] it is shown that every torsion-free group G contains p-basic subgroups for every prime p. Moreover, the p-basic subgroups of G (for the same prime) are all isomorphic.

For each $\lambda \in \Lambda$ (the index set Λ is arbitrary) let $Z(p)_\lambda$ be the group of p-adic integers and Z_λ the infinite cyclic group of finite p-adic integers. Let $P = \sum_{\lambda \in \Lambda}^* Z(p)_\lambda$ be the complete direct sum and $R = \sum_{\lambda \in \Lambda} Z(p)_\lambda$ the discrete direct sum of the groups $Z(p)_\lambda$. If we introduce the n-adic (p-adic) topology for abelian groups (see [9]), then P is complete in the n-adic topology and the n-adic topology coincides with the p-adic topology. R is a pure subgroup of P, hence it possesses a completion in P for the coinciding n-adic and p-adic topologies. Let $C=(\sum_\lambda Z(p)_\lambda)^*$ be the completion of R in P, then, by the remarks of Fuchs, $C=\sum_\lambda' Z(p)_\lambda$ and C is a torsion-free cotorsion group. Moreover C is a direct summand of P.

Let G be an arbitrary torsion-free abelian group. One can define a homomorphism $\sigma: G \to P$ (into) such that the subgroup of elements of infinite p-height in G is the kernel of σ [5]. Assume now that G has no elements of infinite p-height. Then P contains an isomorphic copy $\sigma(G)$ of G. It is known that $\sigma(G)$ is a p-pure subgroup of P and hence $\sigma(G)$ possesses a p-adic completion in P. Let B be a p-basic subgroup of G, then $\sigma(B)$ is a p-basic subgroup of $\sigma(G)$. And $\sigma(B) \leqq \sigma(G)$ implies

(2) p-adic completion of $\sigma(B) \leqq p$-adic completion of $\sigma(G)$.

$\sigma(B)$ is dense in $\sigma(G)$ in the p-adic topology, hence $\sigma(G) \leqq p$-adic completion of $\sigma(B)$, which implies

(3) p-adic completion of $\sigma(G) \leqq p$-adic completion of $\sigma(B)$.

(2) and (3) imply that $\sigma(B)$ and $\sigma(G)$ have identical completions in the p-adic topology. We have proved:

Lemma 1. *Let G be a torsion-free group without elements of infinite p-height. If B is a p-basic subgroup of G, then B and G have identical completions is the p-adic topology.*

If B is a p-basic subgroup of G, then its isomorphic copy $\sigma(B)$ (in $\sigma(G)$) has the form $\sum_{\lambda\in\Lambda} Z_\lambda$ ([5]). As a direct consequence of Lemma 1 we get:

Lemma 2. *$R=\sum_\lambda Z(p)_\lambda$ and $\sigma(B)=\sum_\lambda Z_\lambda$ have the same p-adic completion in P.*

As we have seen the p-adic completion of R has the form $C=\sum_\lambda' Z(p)_\lambda$, so $\sigma(B)$ and $\sigma(G)$ have the same p-adic completion C by the lemma's 1 and 2. Also $\sigma(G)$ is p-pure in P implies $\sigma(G)$ is a p-pure subgroup of C. The torsion-free part of Corollary 2.7 in [5] may be slightly sharpened in the following form:

Theorem 2. *Every torsion-free abelian group G without elements of infinite p-height may be considered to be a p-pure subgroup of some torsion-free cotorsion group $C=\sum_\lambda' Z(p)_\lambda$ and containing $B=\sum_\lambda Z_\lambda$ as a p-basic subgroup. C is the p-adic completion of G and B.*

According to a definition in [6], § 34, p. 114 for p-groups we define a torsion-free group G without elements of infinite p-height to be a *p-closed* group if every Cauchy sequence in G has a limit in G, i.e. if G is complete with respect to the p-adic topology. It is easy now to give results for p-closed groups which are analogous to the corresponding properties of p-groups. Here are 2 examples:

Lemma 3. *The torsion-free group G is p-closed if and only if G is the p-adic completion of a p-basic subgroup B of G (cf. Theorem* 34.1 *in* [6]*).*

Lemma 4. *Two p-closed groups are isomorphic if and only if their p-basic subgroups are isomorphic* (cf. Corollary 34.2, [6], p. 115).

3. Extending homomorphisms. Now we apply the structure theorem 2 to the investigation of torsion-free abelian groups without elements of infinite p-height. We are able to generalize results of Armstrong [1] who obtained extension theorems for homomorphisms of p-pure subgroups of the group of p-adic integers. Let S be a p-pure subgroup of $G=\sum_{\lambda\in\Lambda}' Z(p)_\lambda$ and let S contain $B=\sum_\lambda Z_\lambda$ as a p-basic subgroup. Both G and B are fixed.

Now $\operatorname{Hom}(G/S, G)=0$, since the homomorphic image of a p-divisible group is again p-divisible. But G does not contain p-divisible subgroups $\neq 0$. Then

$$0=\operatorname{Hom}(G/S, G)\to\operatorname{Hom}(G, G)\xrightarrow{j}\operatorname{Hom}(S, G)\to\operatorname{Ext}(G/S, G)\to\operatorname{Ext}(G, G)=0$$

is exact, where $\operatorname{Ext}(G, G)=0$ since G is a cotorsion group. The action of j is to restrict $\pi\in\operatorname{Hom}(G, G)$ to S. Consequently, *every $\alpha\in\operatorname{Hom}(S, G)$ has an extension to an endomorphism of G if and only if* $\operatorname{Ext}(G/S, G)=0$.

In case there exists an extension $\bar\alpha\in\operatorname{Hom}(G, G)$ of $\alpha\in\operatorname{Hom}(S, G)$, then $\bar\alpha$ is uniquely determined, i.e. if $\bar\alpha$, $\bar\beta$ are endomorphisms of G which agree on S, then

$\bar{\alpha}=\bar{\beta}$. For S is contained in the kernel of the difference $\bar{\gamma}=\bar{\alpha}-\bar{\beta}$. Thus $\bar{\gamma}(G)$ is a homomorphic image of the p-divisible group G/S, and, for this reason, is p-divisible. Since G is p-reduced and since $\bar{\gamma}(G)\leqq G$, it follows that $\bar{\gamma}(G)=(\bar{\alpha}-\bar{\beta})(G)=0$. Thus $\bar{\alpha}=\bar{\beta}$.

Lemma 5. *Let L be a subgroup of a torsion-free group H and G an arbitrary torsion-free cotorsion group. Let L_* be the smallest pure subgroup of H containing L and p a rational prime. Then the following are equivalent:*

(1) *L is a p-pure subgroup of H.*

(2) *The p-primary component of the torsion-group L_*/L is* 0.

(3) Ext $(H/L, G)=0$.

(4) Ext $(L_*/L, G)=0$ (cf. Lemma [1], p. 317).

Proof. (1)$\leftrightarrow$(2) ([1], p. 317). Since L_* is pure in H and H is torsion-free, H/L_* is torsion-free. Hence Ext $(H/L_*, G)=0$, since G is cotorsion. Now $0=\mathrm{Ext}(H/L_*, G)\to\mathrm{Ext}(H/L, G)\to\mathrm{Ext}(L_*/L, G)\to 0$ is exact, hence Ext $(H/L, G)=0\leftrightarrow$ Ext $(L_*/L, G)=0$ or (3)$\leftrightarrow$(4). Finally L_*/L is a torsion-group and G is torsion-free, so Ext $(L_*/L, G)\cong\cong$ Hom $(L_*/L, D/G)$, where D is the divisible hull of G. The maximal torsion subgroup T of D/G is a p-group and so Hom $(L_*/L, D/G)=$ Hom $(L_*/L, T)=$ Hom (p-primary component of L_*/L, T), which is zero if and only if the p-primary component of L_*/L is 0, since T is divisible. Hence Ext $(L_*/L, G)=0\leftrightarrow p$-primary component of L_*/L is 0 or (4)$\leftrightarrow$(2). This completes the proof.

Assume again that G is a torsion-free cotorsion group without elements of infinite p-height and let S be a subgroup of G. Then lemma 5 implies, taking $S=L$ and $H=G$, that S is a p-pure subgroup of G if and only if Ext $(G/S, G)=0$. Using our result above about the extension of homomorphisms we get a slight extension of a theorem of Armstrong:

Theorem 3. *Let G be a torsion-free cotorsion group without elements of infinite p-height (p-closed group). Let S be a subgroup of G, then the following are equivalent:*

(i) *S is a p-pure subgroup of G.*

(ii) Ext $(G/S, G)=0$.

(iii) *Every homomorphism of S into G may be extended to an endomorphism of G.* (cf. Theorem, [1], p. 318).

Every torsion-free abelian group S without elements of infinite p-height may be considered to be a p-pure subgroup of a p-closed group by theorem 2, hence such a group satisfies conditions (ii) and (iii).

Now the structure of Hom (G, G) for a p-closed group G can easily be derived. Let $G = \sum_m{}' Z(p)$. We know that $B = \sum_m Z$ is a p-basic subgroup of G. Hence B is a p-pure subgroup of G, but then Hom $(B, G) \cong$ Hom (G, G) by theorem 3. And Hom $(B, G) =$ Hom $(\sum_m Z, G) \cong \sum_m{}^* \text{Hom}(Z, G) \cong \sum_m{}^* G$. Hence Hom $(G, G) \cong \sum_m{}^* G$ Now we are interested is the endomorphism groups of p-pure subgroups of G. First we prove: Let S and T be p-pure subgroups of $G = \sum_{\lambda \in \Lambda}{}' Z(p)_\lambda$. Then each element of Hom (S, T) may be extended uniquely to an endomorphism of G. Indeed, we know that Hom $(S, G) \cong$ Hom (G, G). Hence Hom (S, T) is a subgroup of Hom (G, G). Every $\alpha \in$ Hom (S, T) is a homomorphism of S into G, hence $\alpha \in$ Hom (S, G). But then α has a unique extension $\bar{\alpha}$ to an endomorphism of G.

Next we show: When the elements of Hom (S, T) are identified with their extensions, then Hom (S, T) is a p-pure subgroup of Hom (G, G).

Let $\alpha \in$ Hom (S, T) and identify α with its extension in Hom (G, G). Suppose $\alpha = p^k \mu$, $\mu \in$ Hom (G, G). Then $p^k \mu(a) \in T$ for each $a \in S$. By p-purity of T in G, $\mu(a) \in T$ for each $a \in S$ and therefore $\mu \in$ Hom (S, T).

We have proved the well known

Theorem 4. *Let S and T be p-pure subgroups of a p-closed group G. Then each element of* Hom (S, T) *may be extended uniquely to an endomorphism of G and when the elements of* Hom (S, T) *are identified with their extensions, then* Hom (S, T) *is a p-pure subgroup of* Hom (G, G) (cf. [1], Lemma, p. 139).

Remark. In particular, if S is a p-pure subgroup of $G = \sum_{\lambda \in \Lambda}{}' Z(p)_\lambda$, then each $\alpha \in$ Hom (S, S) may be extended to an endomorphism $\bar{\alpha}$ of G and when we identify α and $\bar{\alpha}$, then Hom (S, S) is a p-pure subgroup of Hom $(G, G) \cong \sum_m{}^* G$, with $|\Lambda| = m$. The question now arises to characterize those p-pure subgroups S of p-closed groups G which have the additional property that Hom $(S, S) \cong \sum_m{}^* S$.

Let S be a p-pure subgroup of $G = \sum_{\lambda \in \Lambda}{}' Z(p)_\lambda$ containing $B = \sum_\lambda Z_\lambda$ as a p-basic subgroup. Both G and B are fixed and to avoid trivialities we suppose that $S \neq B$, $S \neq G$. In order that Hom $(S, S) \cong \sum_\lambda{}^* S$, we must have Hom $(B, S) \cong$ $\cong$ Hom (S, S), since Hom $(B, S) =$ Hom $(\sum_\lambda Z_\lambda, S) \cong \sum_\lambda{}^* \text{Hom}(Z_\lambda, S) \cong \sum_\lambda{}^* S$. Therefore the groups S must have the property that every homomorphism of B into S can be extended to an endomorphism of S.

Now Hom $(S/B, S) = 0$, since S/B is p-divisible, but S is p-reduced. Since B is p-pure in S, we also have Ext $(S/B, G) = 0$, for G is torsion-free cotorsion (lemma 5).

As S/B is p-divisible and G is p-reduced, we get $\mathrm{Hom}\,(S/B, G)=0$. Then

$$0 \to \mathrm{Hom}\,(S/B, S)=0 \to \mathrm{Hom}\,(S/B, G)=0 \to \mathrm{Hom}\,(S/B, G/S) \to \mathrm{Ext}\,(S/B, S) \to$$

$$\to \mathrm{Ext}\,(S/B, G)=0$$

is exact. Hence $\mathrm{Ext}\,(S/B, S) \cong \mathrm{Hom}\,(S/B, G/S)$. Likewise $0=\mathrm{Hom}\,(S/B, S) \to \to \mathrm{Hom}\,(S, S) \xrightarrow{\varphi} \mathrm{Hom}\,(B, S) \to \mathrm{Ext}\,(S/B, S) \to \mathrm{Ext}\,(S, S) \to \mathrm{Ext}\,(B, S)=0$ (B is free) is exact. If φ is *onto*, then we have $\mathrm{Ext}\,(S/B, S) \cong \mathrm{Ext}\,(S, S)$. In other words, if every $\alpha \in \mathrm{Hom}\,(B, S)$ has an extension to an endomorphism of S, then $\mathrm{Ext}\,(S/B, S) \cong \cong \mathrm{Ext}\,(S, S)$. On the other hand, if $\mathrm{Ext}\,(S/B, S) \cong \mathrm{Hom}\,(S/B, G/S)=0$, then every $\alpha \in \mathrm{Hom}\,(B, S)$ has an extension to an endomorphism of S. It can easily be shown that, if $\alpha \in \mathrm{Hom}\,(B, S)$ has an extension $\bar{\alpha} \in \mathrm{Hom}\,(S, S)$, $\bar{\alpha}$ is uniquely determined. We remark also that $\mathrm{Ext}\,(S/B, S)=0$ always implies that $\mathrm{Ext}\,(S, S)=0$. $G = \sum_{\lambda \in \Lambda}' Z(p)_\lambda$ is q-divisible for all primes $q \neq p$, hence G/B, as a homomorphic image of G, is q-divisible for all primes $q \neq p$. But B is a p-basic subgroup of G, so G/B is p-divisible too. Hence *G/B is a divisible group.* Likewise $G/S \cong G/B/S/B$, as a homomorphic image of G/B, *is a divisible group.* Assume that G/S is not torsion-free, then the torsion-part $T \neq 0$ of G/S is a direct summand of G/S, hence $\mathrm{Ext}\,(G/S, G)=0$ implies $\mathrm{Ext}\,(T, G)=0$. It follows that the p-primary component of T is 0 by Lemma 5. So G/S cannot contain elements whose orders are powers of the prime p. In the same way we find that if S/B is a torsion-group, then $\mathrm{Ext}\,(S/B, G)=0$ implies that the p-primary component of S/B is 0. In particular S/B cannot be a p-group.

After these preliminary remarks we now investigate the groups S with

$$\mathrm{Hom}\,(S/B, G/S)=0.$$

We shall need the following

Lemma 6. *Let T be any group and suppose $T \subseteq D$, where D is a divisible group. Then* $\mathrm{Hom}\,(T, D/T)=0$ *implies that T is a divisible group.*

Proof. First we reduce the general case for arbitrary T to the case that T is torsion. If $D=T$, the lemma is trivial. So assume $D/T \neq 0$, and let T_t be the torsion subgroup of T. From $0 \to T_t \to T \to T/T_t \to 0$ is exact it follows that $0 \to \mathrm{Hom}\,(T/T_t, D/T) \to \to \mathrm{Hom}\,(T, D/T)$ is exact. But $\mathrm{Hom}\,(T, D/T)=0$, hence $\mathrm{Hom}\,(T/T_t, D/T)=0$. Suppose that $T/T_t \neq 0$. T/T_t is torsion-free, hence $Z \subseteq T/T_t$. Now $\mathrm{Hom}\,(T/T_t, D/T)=0 \to \to \mathrm{Hom}\,(Z, D/T) \to \mathrm{Ext}\,(T/T_t/Z, D/T)=0$ is exact so $\mathrm{Hom}\,(Z, D/T) \cong D/T=0$, which is a contradiction. Hence $T/T_t=0$ or $T=T_t$. From now on T is supposed to be a torsion group. If $T=0$, the lemma is trivial. Let $T \neq 0$, then T is the direct sum of its p-primary components and since $T \neq 0$, T contains an element of order p_i for some prime p_i. Then T contains a direct summand of type $C(p_i^l)$ ($l \geq 1$ or $l=\infty$) ([6], p. 80).

Suppose $C(p_i^l)$ with finite $l \geqq 1$ is a direct summand of T, then $\operatorname{Hom}(C(p_i^l), D/T)=0$. Since $C(p_i^l)$ is a direct summand of T, $C(p_i^\infty)/C(p_i^l)$ is a direct summand of D/T, hence $\operatorname{Hom}(C(p_i^l), C(p_i^\infty))=0$. This gives a contradiction, since

$$\operatorname{Hom}(C(p_i^l), C(p_i^\infty)) \cong C(p_i^l),$$

as is well known. So, if T contains an element of order p_i, T must contain a direct summand of type $C(p_i^\infty)$. Hence $T = E \oplus T'$, where E is the divisible part ($\neq 0$) of T and T' is the reduced part of T. Suppose T' is not 0. As T' is torsion it contains an element of order p_j for some prime p_j. Then T' contains a direct summand of type $C(p_j^m)$, where $m \geqq 1$ (finite) or $m=\infty$. But T' is reduced, so it cannot contain a direct summand of type $C(p_j^\infty)$. Hence $C(p_j^m)$ is a direct summand of T' for $m \geqq 1$ and finite. But then $C(p_j^m)$ is a direct summand of T which is impossible as we have seen. Hence $T'=0$ and $T=E$ is divisible.

Next we prove:

Lemma 7. $\operatorname{Hom}(S/B, G/S)=0$ *implies that* G/S *is not a torsion-group.*

Proof. $G/S \cong G/B/S/B$, so $\operatorname{Hom}(S/B, G/S) \cong \operatorname{Hom}(S/B, G/B/S/B)=0$ implies that S/B is a divisible group by Lemma 6. Now $\operatorname{Hom}(S/B, G/B/S/B)=0$ implies $\operatorname{Hom}(S/B/(S/B)_t, G/B/S/B)=0$ (see the proof of Lemma 6) implies $S/B/(S/B)_t=0$ or $S/B=(S/B)_t$. It follows that S/B is a divisible torsion-group. Assume now that G/S is a torsion-group. Then S/B torsion and $G/B/S/B \cong G/S$ torsion imply G/B torsion which is a contradiction, since G/B contains $\sum_\lambda Z(p)_\lambda/\Sigma Z_\lambda \cong \Sigma Z(p)_\lambda/Z_\lambda$ as a direct summand and this is a mixed group. Consequently, G/S is not a torsion-group. This completes the proof of Lemma 7.

For the sake of reference we state the next lemma whose proof is contained in the proof of Lemma 7.

Lemma 8. $\operatorname{Hom}(S/B, G/S)=0$ *implies that* S *and* B *have the same (torsion-free) rank and that* S/B *is divisible.*

Remark. By definitions 1. 5 and 1. 6 in [4], p.62 or the remark on p. 45 in [3], the torsion-free groups S with $\operatorname{Hom}(S/B, G/S)=0$ are *quotient-divisible* groups, as S contains a free group B and S/B is a divisible torsion-group.

Now we prove:

Theorem 5. *Let S be a torsion-free group without elements of infinite p-height and with rank $\leqq \aleph_0$. Consider S as a p-pure subgroup of $G = \sum'_{\lambda \in \Lambda} Z(p)_\lambda$, while S contains $B = \sum_{\lambda \in \Lambda} Z_\lambda$ as a p-basic subgroup. Then the following are equivalent:*

(i) $\operatorname{Ext}(S/B, S)=0$ *(or* $\operatorname{Hom}(S/B, G/S)=0$*).*

(ii) *S and B have the same torsion-free rank and* $\operatorname{Ext}(S, S)=0$.

(iii) *There exists a torsion-free quotient-divisible group* S' *of rank* 1 *with* $S' \subseteqq Q_p$, *such that* $S \cong \sum_{\lambda \in \Lambda} S'_\lambda$. *($Q_p$ is the group of all rationals with denominators prime to p).*

Proof. (i)$\to$(ii) is clear from lemma 8 and the preliminary remarks of lemma 6.

(ii)$\to$(iii). Since rank $S \leqq \aleph_0$ we can apply lemma 4. 2 of J. HAUSEN in ([10], p. 170) which assures us of the existence of a group S' of rank 1, torsion-free and quotient-divisible, such that $S \cong \sum S'$ (direct sum). Since S' has rank 1 and S and B have the same rank (by (ii)) we must have $S \cong \sum_{\lambda \in \Lambda} S'_\lambda (|\Lambda| = \text{rank } S = \text{rank } B)$. Now S has no elements of infinite p-height, so S' (as a direct summand) has the same property. Then $S' \subseteqq Q_p$.

(iii)$\to$(i). From $S \cong \sum_\lambda S', B = \sum_\lambda Z$ we infer that $S/B \cong \sum_\lambda S'/Z \cong \sum_\lambda \sum_{t \in P} C(t^\infty)$, where P is a set of primes and $p \notin P$, since $S' \subseteqq Q_p$. Now $\operatorname{Ext}(S/B, S) \cong$ $\cong \operatorname{Hom}(S/B, D/S)$, where D is the divisible hull of S ([6], p. 244). Since rank $S=$ $=\text{rank } B = |\Lambda|$, we get $D = \sum_\lambda Q$. Hence $D/B \cong \sum_\lambda Q/Z \cong \sum_\lambda \sum_s C(s^\infty)$, where the summation $\sum_s$ is taken over all primes s. Then $D/S \cong D/B/S/B \cong$ $\cong \sum_\lambda (\sum_s C(s^\infty)/\sum_{t \in P} C(t^\infty)) \cong \sum_\lambda (\sum_{u \in C(P)} C(u^\infty))$, where $C(P)$ is the complement of P in the set of all primes. Then

$$\operatorname{Hom}(S/B, D/S) \cong \operatorname{Hom}\Big(\sum_\lambda \sum_{t \in P} C(t^\infty), \sum_\lambda \sum_{u \in C(P)} C(u^\infty)\Big) = 0.$$

This completes the proof of theorem 5.

It may be remarked that each of the conditions (i), (ii) and (iii) is sufficient in order that every $\alpha \in \operatorname{Hom}(B, S)$ may be extended uniquely to an endomorphism of S. Now we specialize to the case of finite rank. We recall that a non-nil group of rank 1 is a torsion-free group of rank 1 with characteristic $(k_1, k_2, \ldots, k_i, \ldots)$ with either $k_i=0$ or $k_i=\infty$ for all i. The quotient-divisible groups of rank 1 are exactly the non-nil groups of rank 1.

Theorem 6. *Let n be a natural number* $\geqq 1$. *Let S be a torsion-free group and a proper p-pure subgroup of* $G = \sum_1^n Z(p)$ *while S contains* $B = \sum_1^n Z$ *as a p-basic subgroup. Then the following are equivalent:*

(i) $\operatorname{Hom}(S/B, G/S)=0$ *(or* $\operatorname{Ext}(S/B, S)=0$*).*

(ii) *S has rank n and* $\operatorname{Ext}(S, S)=0$.

(iii) *S has rank n and every* $\alpha \in \mathrm{Hom}(B, S)$ *may be extended to an endomorphism of S.*

(iv) *S is isomorphic to the direct sum of n isomorphic non-nil groups* S' *of rank* 1 *with* $S' \subseteqq Q_p$, *where* Q_p *is the group of all rationals with denominators prime to p.*

Clearly (i)$\leftrightarrow$(ii)$\leftrightarrow$(iv) by Theorem 5. That (iii)$\leftrightarrow$(iv) is a special case of the next result. We now investigate the p-pure subgroups S of $G = \sum_1^n Z(p)$ containing $B = \sum_1^n Z$ as a p-basic subgroup and with the property that every $\alpha \in \mathrm{Hom}(B, S)$ may be extended to an endomorphism of S. We do not assume that $\mathrm{Hom}(S/B, G/S)=0$.

Theorem 7. *Let S be a p-pure subgroup of* $G = \sum_1^n Z(p)$ *containing* $B = \sum_1^n Z$ *as a p-basic subgroup. Then the following are equivalent:*

(i) *Every* $\alpha \in \mathrm{Hom}(B, S)$ *may be extended to an endomorphism of S.*

(ii) *S is isomorphic to the direct sum of n isomorphic groups I, such that I is a subgroup of* $Z(p)$ *which contains* 1 *and with the property* $\pi I \subseteqq I$ *for any* $\pi \in I$.

Proof. (i)$\rightarrow$(ii). Since S is p-pure in $G = \sum_1^n Z(p)$, every $\delta \in \mathrm{End}\, S$ has a unique extension $\bar{\delta} \in \mathrm{End}\, G$. So each element $\delta \in \mathrm{End}\, S$ is a (left) multiplication endomorphism by an $n \times n$-matrix with entries in $Z(p)$. Since $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots$ $\ldots, (0, 0, \ldots, 1) \in S$ the columns in the $n \times n$-matrix are elements of S. Now $\mathrm{Hom}(S, S) = \mathrm{Hom}(B, S) \cong \sum_1^n S$, so any $(\pi_1, \pi_2, \ldots, \pi_n) \in \sum_1^n S (\pi_i \in S)$ may be used as a multiplicator on the left, inducing an endomorphism of S, in other words,

$$\begin{pmatrix} \pi_{11} & \cdots & \pi_{1n} \\ \cdots & \cdots & \cdots \\ \pi_{n1} & \cdots & \pi_{nn} \end{pmatrix} S \leqq S,$$ whenever the columns are elements of S. Then

$$\begin{pmatrix} 1 & 0 \ldots 0 \\ 0 & 0 \ldots 0 \\ \cdots & \cdots \\ 0 & 0 \ldots 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_1 \\ \cdot \\ \pi_n \end{pmatrix} = \begin{pmatrix} \pi_1 \\ 0 \\ \cdot \\ 0 \end{pmatrix} \in S \quad \text{if} \quad \begin{pmatrix} \pi_1 \\ \pi_2 \\ \cdot \\ \pi_n \end{pmatrix} \in S.$$

Similar for other components. Hence in S we have the direct-sum decompositon:

$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \cdot \\ \pi_n \end{pmatrix} = \begin{pmatrix} \pi_1 \\ 0 \\ \cdot \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \pi_2 \\ \cdot \\ 0 \end{pmatrix} + \ldots + \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \pi_n \end{pmatrix}.$$

The elements of the form $\begin{pmatrix} 0 \\ \cdot \\ \pi_j \\ 0 \end{pmatrix} \in S$ form the subgroup I_j in S. Then $S = I_1 \oplus \cdots \oplus I_n$.

If we identify $\begin{pmatrix} 0 \\ \cdot \\ \pi_j \\ 0 \end{pmatrix} \leftrightarrow \pi_j$, then each I_j is a subgroup of $Z(p)$. As a direct summand, I_j is a pure, hence p-pure, subgroup of S. S is p-pure in $G = \sum_1^n Z(p)$, so I_j is p-pure in $Z(p)$. Hence every map of I_j into I_k is the restriction of an endomorphism of $Z(p)$ (theorem 4), i.e. every map of I_j into I_k is a (left) multiplication by an element $\pi \in Z(p)$. Since $1 \in I_j$, $\pi.\ 1 = \pi \in I_k$. Then $\operatorname{Hom}(S, S) = \operatorname{Hom}(I_1 \oplus \cdots \oplus I_n, I_1 \oplus \cdots \oplus I_n) \cong$ $\cong \sum_{j,k} \operatorname{Hom}(I_j, I_k)$ and $\operatorname{Hom}(B, S) \cong \sum_1^n S = \sum_1^n (I_1 \oplus \cdots \oplus I_n)$ and every map in $\operatorname{Hom}(B, S)$ is the restriction of a map in $\operatorname{Hom}(S, S)$ imply $\operatorname{Hom}(I_j, I_k) \cong I_k$ $(j, k = 1, \ldots, n)$. Then $I_k I_j \subseteqq I_k$, but $\pi.\ 1 = \pi \in I_k$ for any $\pi \in I_k$ implies $I_k I_j = I_k$. Since $I_k I_j = I_j I_k$ it follows that $I_j = I_k$ $(j, k = 1, \ldots, n)$. So, if we put $I_j = I$, we get $S = I \oplus I \oplus$ $\oplus \cdots \oplus I$ (n summands). Moreover I is a subgroup of $Z(p)$ with $II = I$ or $\pi I \subseteqq I$ for any $\pi \in I$.

(ii) $\to$ (i). $S = \sum_1^n I$, where I is a subgroup of $Z(p)$ with $\pi I \subseteqq I$ for any $\pi \in I$. I is p-pure in S, S is p-pure in $\sum_1^n Z(p)$, so I is p-pure in $\sum_1^n Z(p)$, hence I is p-pure in $Z(p)$. So each $\alpha \in \operatorname{End} I$ has a unique extension to an endomorphism of $Z(p)$. Then each element of $\operatorname{End} I$ is a left multiplication endomorphism by an element $\pi \in Z(p)$. Since $1 \in I$, $\pi.\ 1 = \pi$ is in I. So $\operatorname{End} I \subseteqq L_I$, where L_I denotes the set of all left multiplication endomorphism by the elements of I. Then $\operatorname{End} I \subseteqq L_I$ and I, as a ring, is a subring of $Z(p)$ imply $\operatorname{End} I = L_I$ (Lemma, [1], p. 319), in other words $\operatorname{Hom}(I, I) \cong I$. From $\operatorname{Hom}(S, S) \cong \sum_1^{n^2} \operatorname{Hom}(I, I)$ and

$$\operatorname{Hom}(B, S) \cong \sum_1^{n^2} \operatorname{Hom}(Z, I)$$

we infer that every map of $\operatorname{Hom}(B, S)$ may be extended to an endomorphism of S. This completes the proof of Theorem 7.

Remark. If S has rank n, then (i) resp. (ii) of Theorem 7 pass into (iii) resp. (iv) of Theorem 6.

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Remarks on inequalities of series of positive terms

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We proved the following inequalities for non-negative a_n and b_n:

$$\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} a_k^\gamma b_{n-k}^\gamma\right)^{1/\gamma} \geqq 1\left(\sum_{k=-\infty}^{\infty} a_k^r\right)^{1/r}\left(\sum_{k=-\infty}^{\infty} b_k^s\right)^{1/s}, \quad {}^{*)} \tag{1}$$

where $1 \leqq r, s, \gamma \leqq \infty$ and $\frac{1}{r}+\frac{1}{s} = 1+\frac{1}{\gamma}$ (see Theorem in [1]); and

$$(\sup_i a_i)(\sup_i b_i)+\sum_{n=-\infty}^{\infty} \sup_k a_k b_{n-k} \geqq p^{1/p} q^{1/q}\left(\sum_{k=-\infty}^{\infty} a_k^p\right)^{1/p}\left(\sum_{k=-\infty}^{\infty} b_k^q\right)^{1/q}, \tag{2}$$

where $\frac{1}{p}+\frac{1}{q} = 1$ (see Theorem 1 in [2]).

We also formulated in [2], without proof, the following inequality. If $\{a_n^{(i)}\}$ ($i=1, 2, \dots, m$; $n=0, \pm 1, \pm 2, \dots$) are m sequences of non-negative numbers, then

$$(m-1)\prod_{i=1}^{m}(\sup_k a_k^{(i)})+\sum_{n=-\infty}^{\infty}\sup_{k_1+k_2+\dots+k_m=n} a_{k_1}^{(1)} a_{k_2}^{(2)} \cdots a_{k_m}^{(m)} \geqq \tag{3}$$

$$\geqq \prod_{i=1}^{m}(p_i)^{1/p_i}\prod_{i=1}^{m}\left(\sum_{k=-\infty}^{\infty}(a_k^{(i)})^{p_i}\right)^{1/p_i},$$

where $1 \leqq p_i \leqq \infty$ and $\sum_{i=1}^{m}\frac{1}{p_i}=1$.

In the present note *we show that the constant factors* 1, $p^{\frac{1}{p}} q^{\frac{1}{q}}$, *and* $\prod_{i=1}^{m}(p_i)^{\frac{1}{p_i}}$ *in* (1), (2), *and* (3), *respectively, are best possible.*

Since inequality (2) is a special case of (3), it would be sufficient to prove that the constant factor in (3) is best possible. In spite of this we prove both cases, because the idea of proof can be seen much better in the simple case, furthermore the proof of the general case is not a trivial straightforward generalization.

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*) If $\gamma=\infty$ and $C_k \geqq 0$, then $\left\{\sum_{n=-\infty}^{\infty} C_n^\gamma\right\}^{1/\gamma}$ means $\sup_n C_n$.

Moreover, with respect to the obvious inequality

$$\left(\sum_{k=-\infty}^{\infty} a_k^\gamma b_{n-k}^\gamma\right)^{1/\gamma} \geqq \sup_k a_k b_{n-k},$$

which holds for any positive γ and n, we remark that (2) implies the following

Corollary. *Suppose that $\gamma>0$, $1\leqq p, q\leqq\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then we have*

$$(\sup_n a_n)(\sup_n b_n)+\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} a_k^\gamma b_{n-k}^\gamma\right)^{1/\gamma} \geqq \tag{4}$$
$$\geqq p^{1/p}q^{1/q}\left(\sum_{n=-\infty}^{\infty} a_n^p\right)^{1/p}\left(\sum_{n=-\infty}^{\infty} b_n^q\right)^{1/q}.$$

The factor $p^{\frac{1}{p}}q^{\frac{1}{q}}$ in (4) seems not to be best possible generally. But if $p=q=2$ then it is best possible for any $\gamma>0$. Namely, if $a_0=b_0=1$ and $a_n=b_n=0$ if $n\neq 0$, then both sides of (4) equal 2.

Setting the above given sequences $\{a_n\}$ and $\{b_n\}$ into (1) we obtain an equality. This verifies that the factor 1 in (1) is best possible.

The proof of the fact that the factor $p^{\frac{1}{p}}q^{\frac{1}{q}}$ in (2) is best possible is a little bit longer.

It is easy to see that if $a_0=a_1=\cdots=a_{\nu-1}=1$, $b_0=b_1=\cdots=b_{\mu-1}=1$ and $a_n=b_n=0$ otherwise, then inequality (2) reduces to

$$\nu+\mu\geqq p^{1/p}q^{1/q}\nu^{1/p}\mu^{1/q}. \tag{5}$$

If we can show that for any positive ε there exist integers ν and μ such that

$$\nu+\mu<(p^{1/p}q^{1/q}+\varepsilon)\nu^{1/p}\mu^{1/q}, \tag{6}$$

then our statement will be proved.

Inequality (6) is equivalent to

$$\left(\frac{\nu}{\mu}\right)^{1/q}+\left(\frac{\mu}{\nu}\right)^{1/p}<p^{1/p}q^{1/q}+\varepsilon \quad \left(\frac{1}{p}+\frac{1}{q}=1\right). \tag{7}$$

If $p=1$ and $q=\infty$ then (7) means

$$1+\frac{\mu}{\nu}<1+\varepsilon,$$

and this obviously follows if $\mu=1$ and ν is large enough.

The case $q=1$ and $p=\infty$ can be verified similarly.

If $1<p, q<\infty$ then we have

$$\left(\frac{q}{p}\right)^{1/q}+\left(\frac{p}{q}\right)^{1/p}=p^{1/p}q^{1/q}\left(\frac{1}{p^{1/p+1/q}}+\frac{1}{q^{1/p+1/q}}\right)=p^{1/p}q^{1/q}. \tag{8}$$

Since the functions $y=x^\alpha$ $(\alpha>0)$ are continuous at any point $x_0>0$, by (8) we have that

$$x^{1/q}+\left(\frac{1}{x}\right)^{1/p}\to p^{1/p}q^{1/q} \text{ as } x\to\frac{q}{p}.$$

Consequently if $x=\frac{\nu}{\mu}$ approximates the value $\frac{q}{p}$ "in a suitable way", then (7) and (6) hold. This proves that the factor $p^{\frac{1}{p}}q^{\frac{1}{q}}$ in (2) is best possible.

To prove that the factor $\prod_{i=1}^{m}(p_i)^{\frac{1}{p_i}}$ in (3) is best possible, we start with the case $p_1=1$ and $p_2=\dots=p_m=\infty$.

Set $a_0^{(i)}=a_1^{(i)}=\dots=a_{\nu_i-1}^{(i)}=1$ and $a_k^{(i)}=0$ otherwise, for $i=1, 2, \dots, m$. Then inequality (3) reduces to

$$m-1+1+\sum_{i=1}^{m}(\nu_i-1)\geqq 1\cdot\nu_1 .$$

Hence, if $\nu_2=\nu_3=\dots=\nu_m=1$ and ν_1 is large enough, it can be seen that $1\left(=\prod_{i=1}^{m}(p_i)^{\frac{1}{p_i}}\right)$ is best possible indeed.

If $1<p_1, p_2, \dots, p_\mu<\infty$ and $p_{\mu+1}=\dots=p_m=\infty$, then inequality (3) reduces to

$$(m-1)+1+\sum_{i=1}^{m}(\nu_i-1)\geqq\prod_{i=1}^{\mu}(p_i)^{1/p_i}\prod_{i=1}^{\mu}(\nu_i)^{1/p_i}. \tag{9}$$

Since $\sum_{i=1}^{m}\frac{1}{p_i}=\sum_{i=1}^{\mu}\frac{1}{p_i}=1$ we have

$$\sum_{j=1}^{\mu}\prod_{i=1}^{\mu}\left(\frac{p_i}{p_j}\right)^{1/p_i}=\prod_{i=1}^{\mu}(p_i)^{1/p_i}\sum_{j=1}^{\mu}\frac{1}{p_j^{\sum_{i=1}^{\mu}1/p_i}}=\prod_{i=1}^{\mu}(p_i)^{1/p_i}. \tag{10}$$

By (10) it can be seen that for any $\varepsilon>0$ there exist rational numbers r_i $(i=1, \dots, \mu)$ such that

$$\sum_{j=1}^{\mu}\prod_{i=1}^{\mu}\left(\frac{r_i}{r_j}\right)^{1/p_i}<\prod_{i=1}^{\mu}(p_i)^{1/p_i}+\varepsilon. \tag{11}$$

Choose the positive integers $l_1, \cdots, l_\mu$ such that

$$\frac{r_1}{r_2}=\frac{l_2}{l_1},\quad \frac{r_1}{r_3}=\frac{l_3}{l_1},\ \cdots,\ \frac{r_1}{r_\mu}=\frac{l_\mu}{l_1},$$

that is,

$$r_1l_1=r_2l_2=r_3l_3=\dots=r_\mu l_\mu . \tag{12}$$

Define $v_1=l_1N$, $v_2=l_2N$, ..., $v_\mu=l_\mu N$ and $v_{\mu+1}=\cdots=v_m=1$, where N is a natural number to be defined later. Then inequality (9) has the form

$$m+\sum_{i=1}^{\mu}(Nl_i-1)\geqq\prod_{i=1}^{\mu}(p_i)^{1/p_i}\prod_{i=1}^{\mu}(Nl_i)^{1/p_i}.$$

We show that if N is large enough then

$$m+\sum_{i=1}^{\mu}(Nl_i-1)<\left(\prod_{i=1}^{\mu}(p_i)^{1/p_i}+\varepsilon\right)\prod_{i=1}^{\mu}(Nl_i)^{1/p_i}, \tag{13}$$

which verifies that the factor $\prod_{i=1}^{m}(p_i)^{\frac{1}{p_i}}$ is best possible. By (12),

$$\frac{m-\mu+N\sum_{j=1}^{\mu}l_j}{N\prod_{i=1}^{\mu}(l_i)^{1/p_i}}<\frac{m-\mu}{N}+\sum_{j=1}^{\mu}\frac{l_j^{\sum_{i=1}^{\mu}1/p_i}}{\prod_{i=1}^{\mu}(l_i)^{1/p_i}}=\frac{m-\mu}{N}+\sum_{i=1}^{\mu}\prod_{i=1}^{\mu}\left(\frac{l_j}{l_i}\right)^{1/p_i}=$$

$$=\frac{m-\mu}{N}+\sum_{j=1}^{\mu}\prod_{i=1}^{\mu}\left(\frac{r_i}{r_j}\right)^{1/p_i}.$$

Hence, if N is large enough we obtain by (11) that

$$\frac{m-\mu+N\sum_{j=1}^{\mu}l_j}{N\prod_{i=1}^{\mu}(l_i)^{1/p_i}}<\prod_{i=1}^{\mu}(p_i)^{1/p_i}+\varepsilon.$$

So we have proved (13), and this completes our proof.

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Function algebras and flows

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§ **1.** Throughout this article X will denote a fixed compact Hausdorff space upon which the real line $\mathbf{R}$ (with the usual topology) acts as a locally compact transformation group. The pair $(X, \mathbf{R})$ will be called a *flow* and the translate of an x in X by a t in $\mathbf{R}$ will be written $x+t$. The space of all continuous complex-valued functions on X will be denoted by $C(X)$. If φ is a function in $C(X)$, then φ will be called *analytic* in case for each x in X the function $\varphi(x+t)$ of t is the boundary function of a function which is bounded and analytic in the upper half-plane. The space of all analytic functions in $C(X)$ will be denoted by $\mathfrak{A}$. It is clear that $\mathfrak{A}$ is a uniformly closed subalgebra of $C(X)$ which contains the constant functions.

This notion of analyticity was first defined by FORELLI and has been studied extensively by him in a number of articles (see [1], [2], [3], [4] and [5]). Our principal objective in this article is to show that under suitable conditions the algebra $\mathfrak{A}$ belongs to well known classes of abstract function algebras.

Recall that if $\mathfrak{B}$ is an algebra of continuous functions on a compact Hausdorff space Y then a probability measure m on Y is called a *representing measure* for $\mathfrak{B}$ in case $\int_Y \varphi\psi\, dm = \left(\int_Y \varphi\, dm\right)\left(\int_Y \psi\, dm\right)$ for all φ and ψ in $\mathfrak{B}$. If m is a representing measure for $\mathfrak{B}$ and if $\mathfrak{B}$ contains the constant functions, then $\mathfrak{B}$ is called a *weak-$*$ Dirichlet algebra* in $L^\infty(m)$ in case $\mathfrak{B}+\overline{\mathfrak{B}}$ is weak-$*$ dense in $L^\infty(m)$. (The bar denotes conjugation, here and always.) We refer the reader to [16] for an account of weak-$*$ Dirichlet algebras. If $\mathfrak{B}$ contains the constant functions and if $\mathfrak{B}+\overline{\mathfrak{B}}$ is uniformly dense in the space of continuous functions on Y, then $\mathfrak{B}$ is called a *Dirichlet algebra*.

Our first basic structure theorem is

Theorem I. *If μ is an invariant, ergodic, probability measure on X, then μ is a representing measure for $\mathfrak{A}$ and $\mathfrak{A}$ is a weak-$*$ Dirichlet algebra in $L^\infty(\mu)$.*

With an additional hypothesis on the flow $(X, \mathbf{R})$ er are able to prove a much stronger theorem. The hypothesis is that $(X, \mathbf{R})$ is *strictly* (or *uniquely*) *ergodic*

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in the sense that there is exactly one probability measure on X which is invariant under the action of $\mathbf{R}$. It is well known that since X is compact there is at least one invariant probability measure on X. However, the requirement that there is exactly one is very special (see [6], [11], and [14]). Note that if $(X, \mathbf{R})$ is strictly ergodic, then the unique invariant probability measure must be ergodic and this explains the terminology.

Our second basic structure theorem is

Theorem II. *If the flow* $(X, \mathbf{R})$ *is strictly ergodic, then* $\mathfrak{A}$ *is a Dirichlet algebra on* X.

Theorems I and II enable us to exhibit new and striking ways in which algebras of analytic functions associated with flows generalize certain spaces of generalized analytic functions in the sense of Arens and Singer. To see this recall how these spaces are defined. Let Γ be a non-zero subgroup of $\mathbf{R}$, give Γ the discrete topology, and let G be its compact character group. The space $\mathfrak{A}_\Gamma$ of generalized analytic functions determined by Γ is defined to be the space of all continuous complex-valued functions on G whose Fourier transforms vanish on the negative half of Γ. It is easy to see that $\mathfrak{A}_\Gamma$ is a Dirichlet algebra on G. If Γ is (isomorphic to) the integers, then $\mathfrak{A}_\Gamma$ is simply the classic disc algebra. In general $\mathfrak{A}_\Gamma$ may be regarded as the algebra of analytic almost periodic functions on the line whose frequencies lie in Γ. The real line can be imbedded in G as the space of all characters on Γ which are continuous with respect to the usual topology of $\mathbf{R}$ restricted to Γ. The imbedding defines a natural action of $\mathbf{R}$ on G so that $(G, \mathbf{R})$ is a flow. It is not hard to see that the space of analytic functions associated with this flow is precisely $\mathfrak{A}_\Gamma$. The flow $(G, \mathbf{R})$ is strictly ergodic — normalized Haar measure on G is the unique probability measure on G which is invariant under the action of $\mathbf{R}$. Thus Theorem II gives a new, albeit roundabout, proof that $\mathfrak{A}_\Gamma$ is a Dirichlet algebra. There are many flows, even strictly ergodic ones, which are not of the form $(G, \mathbf{R})$ and consequently, the results which we obtain for spaces of analytic functions associated with general flows represent bona fide extensions of results known to hold for spaces of generalized analytic functions.

Section 2 is devoted to the proofs of Theorems I and II. In section 3 we investigate the nature of general representing measures for $\mathfrak{A}$. Our investigation is incomplete in some respects; however, it is sufficiently complete to yield results which are complementary to Forelli's generalization of the F. and M. Riesz Theorem [1]. Specifically, we present conditions on a representing measure m under which the abstract Hardy spaces $H^p(m)$, $1 \leqq p \leqq \infty$, have the property that no non-zero function in $H^p(m)$ can vanish on a set of positive measure. These conditions are satisfied in the following two situations:

(i) $(X, \mathbf{R})$ is an arbitrary flow and m is invariant and ergodic.

(ii) $(X, \mathbf{R})$ is strictly ergodic, and m is an arbitrary representing measure for $\mathfrak{A}$. This feature of the Hardy spaces is well known when $\mathfrak{A}$ is the disc algebra. In [9] HELSON and LOWDENSLAGER showed that if $\mathfrak{A}_\Gamma$ is the algebra of generalized analytic functions associated with the subgroup Γ of $\mathbf{R}$, and if m is Haar measure on G then the Hardy spaces $H^p(m)$, $1 \leqq p \leqq \infty$, also have this feature. In the situations (i) and (ii), we also show that $H^\infty(m)$ is a maximal weak-$*$ closed subalgebra of $L^\infty(m)$. This result is complementary to FORELLI's generalization [5] of WERMER's maximality theorem.

§ 2. The dual space of $C(X)$ is the space of all bounded, complex, Baire measures on X; we will denote it by $M(X)$. If φ is in $C(X)$ and if λ is in $M(X)$, the integral $\int_X \varphi \, d\lambda$ will often be written as $\langle \varphi, \lambda \rangle$.

The definition of analyticity given in section 1, while intuitively appealing, is not the one which we shall use in our proofs. We digress momentarily in order to give an equivalent definition.

The action of $\mathbf{R}$ on X induces a strongly continuous, one-parameter group $\{T_t\}_{t \in \mathbf{R}}$ of automorphisms of $C(X)$. These are defined by the formula

$$(T_t \varphi)(x) = \varphi(x-t), \quad \varphi \in C(X), \quad t \in \mathbf{R}.$$

Using $\{T_t\}_{t \in \mathbf{R}}$ one may convolve a function in $C(X)$ or a measure in $M(X)$ with a function in $L^1(\mathbf{R})$ in the following way. For φ in $C(X)$ and f in $L^1(\mathbf{R})$, $\varphi * f$ is defined by setting

$$\varphi * f = \int_{-\infty}^{\infty} (T_t \varphi) f(t) \, dt.$$

If λ is in $M(X)$ and if f is in $L^1(\mathbf{R})$, $\lambda * f$ is defined to be the measure such that

$$\langle \varphi, \lambda * f \rangle = \langle \varphi * \tilde{f}, \lambda \rangle$$

for all φ in $C(X)$ where $\tilde{f}$ is the function whose value at t in $\mathbf{R}$ is $f(-t)$. Under these operations of convolution $C(X)$ and $M(X)$ become $L^1(\mathbf{R})$-modules such that

$$\|\varphi * f\| \leqq \|\varphi\| \, \|f\| \quad \text{and} \quad \|\lambda * f\| \leqq \|\lambda\| \, \|f\|$$

for all φ in $C(X)$, λ in $M(X)$, and f in $L^1(\mathbf{R})$. For each φ in $C(X)$ (resp., λ in $M(X)$) let $J(\varphi)$ (resp., $J(\lambda)$) be $\{f \in L^1(\mathbf{R}) | \varphi * f = 0\}$ (resp., $\{f \in L^1(\mathbf{R}) | \lambda * f = 0\}$). Then $J(\varphi)$ (resp., $J(\lambda)$) is an ideal in $L^1(\mathbf{R})$ which is closed by the above inequalities. The hull of $J(\varphi)$ (resp., $J(\lambda)$) is defined to be the *spectrum* of φ (resp., λ) in the sense of spectral synthesis and will be denoted by $\operatorname{sp}(\varphi)$ (resp., $\operatorname{sp}(\lambda)$). (Recall that the hull of an ideal in $L^1(\mathbf{R})$ is by definition the intersection of the zero sets of the Fourier transforms of the functions in the ideal.) We refer the reader to [1] for an account of spectra.

The equivalent formulation of analyticity is given in Proposition 2. 1 below. It was used by FORELLI in [5] although he never formally stated or proved it. The proof is an easy calculation based on the well known fact that a bounded measurable function F on $\mathbf{R}$ is the boundary function of a function which is analytic and bounded in the upper half-plane if and only if the spectrum of F in the usual sense of spectral synthesis is non-negative. Because of this, the proof will not be given.

Proposition 2. 1. *A function* φ *in* $C(X)$ *is analytic if and only if* $\mathrm{sp}(\varphi)\subseteqq[0, \infty)$.

Following FORELLI [1] we shall call a measure in $M(X)$ *analytic* in case its spectrum is non-negative.

The proofs of Theorems I and II rest on

Proposition 2. 2. *A measure* λ *in* $M(X)$ *is invariant under the action of* $\mathbf{R}$ *if and only if* $\mathrm{sp}(\lambda)$ *is contained in the singleton* $\{0\}$.

Proof. Suppose λ is invariant. If φ is in $C(X)$ and if f is in $L^1(\mathbf{R})$, then by Fubini's Theorem we obtain the equation

$$\langle\varphi, \lambda * f\rangle=\langle\varphi * \hat{f}, \lambda\rangle=\int_{-\infty}^{\infty}\langle T_t\varphi, \lambda\rangle\hat{f}(t)\,dt=\langle\varphi, \lambda\rangle\int_{-\infty}^{\infty} f(t)\,dt=\langle\varphi, \lambda\rangle\hat{f}(0)$$

where $\hat{f}$ is the Fourier transform of f. It follows easily that $\mathrm{sp}(\lambda)\subseteqq\{0\}$.

Suppose, conversely, that $\mathrm{sp}(\lambda)\subseteqq\{0\}$. Choose φ in $C(X)$ and for t in $\mathbf{R}$ let $F(t)=\langle T_t\varphi, \lambda\rangle$. On page 50 of [1] FORELLI showed that the spectrum of F as a bounded continuous function is contained in $-\mathrm{sp}(\varphi)\cap\mathrm{sp}(\lambda)\subseteqq\{0\}$. By [15, 7. 8. 3 (e)], F is constant. Thus for all φ in $C(X)$ and all t in $\mathbf{R}$, $\langle T_t\varphi, \lambda\rangle=\langle\varphi, \lambda\rangle$, and consequently, λ is invariant.

Proof of Theorem I:

(i) Let $\mathfrak{A}_\mu=\{\varphi\in\mathfrak{A}|\langle\varphi, \mu\rangle=0\}$. To show that μ is a representing measure for $\mathfrak{A}$ it clearly suffices to show that $\mathfrak{A}_\mu$ is an ideal in $\mathfrak{A}$. To this end, let $\mathfrak{A}_0$ be the intersection of $\mathfrak{A}$ with the weak-$*$ closure in $L^\infty(\mu)$ of the space of all functions φ in $\mathfrak{A}$ such that $\mathrm{sp}(\varphi)\subseteqq(0, \infty)$. Then by Lemma 3 and Theorem 1 of [1] it is easy to see that $\mathfrak{A}_0$ is a closed ideal in $\mathfrak{A}$. We show that $\mathfrak{A}_\mu=\mathfrak{A}_0$. Since μ is invariant, $\mathrm{sp}(\mu)\subseteqq\{0\}$ by Proposition 2. 2, and so μ is analytic. Hence, by Proposition 2 of [1], $\mathfrak{A}_0\subseteqq\mathfrak{A}_\mu$. Suppose φ is a function in $\mathfrak{A}_\mu$ which is not in $\mathfrak{A}_0$. Then by the Hahn—Banach Theorem there is a function f in $L^1(\mu)$ such that $\langle\varphi, f d\mu\rangle=1$ while $\langle\psi, f d\mu\rangle=0$ for all ψ in $\mathfrak{A}_0$. Observe that since $f d\mu$ annihilates $\mathfrak{A}_0$, $f d\mu$ is an analytic measure by Proposition 2 of [1]. For t in $\mathbf{R}$, let $F(t)=\langle T_t\varphi, f d\mu\rangle$. Then as in the proof of Proposition 2. 2, we find that $\mathrm{sp}(F)\subseteqq-\mathrm{sp}(\varphi)\cap\mathrm{sp}(f d\mu)\subseteqq(-\infty, 0)\cap[0, \infty)=\{0\}$, and so F is constant. The constant is $\langle\varphi, f d\mu\rangle=1$. For positive τ, let

$\varphi_\tau = \frac{1}{2\tau} \int_{-\tau}^{\tau} (T_t\varphi)\,dt$. Then by the individual ergodic theorem [8, p. 18] and the fact that μ is ergodic, the φ_τ converge a.e. (μ) to $\langle \varphi, \mu \rangle = 0$. The convergence of the φ_τ is also bounded and so we find that

$$\lim_{\tau \to \infty} \langle \varphi_\tau, f\,d\mu \rangle = 0.$$

On the other hand, by Fubini's Theorem,

$$\langle \varphi_\tau, f\,d\mu \rangle = \frac{1}{2\tau} \int_{-\tau}^{\tau} \langle T_t \varphi, f\,d\mu \rangle = \frac{1}{2\tau} \int_{-\tau}^{\tau} F(t)\,dt = 1$$

for all positive τ. Thus

$$\lim_{\tau \to \infty} \langle \varphi_\tau, f\,d\mu \rangle = 1.$$

This contradiction shows that $\mathfrak{A}_\mu = \mathfrak{A}_0$ as we promised and the proof of the first half of Theorem I is complete.

(ii) To show that $\mathfrak{A} + \overline{\mathfrak{A}}$ is weak-$*$ dense in $L^\infty(\mu)$ suppose f is a function in $L^1(\mu)$ which annihilates $\mathfrak{A} + \overline{\mathfrak{A}}$. By Proposition 2′ of [1] and the fact that

$$\overline{\mathfrak{A}} = \{\varphi \in C(X) | \operatorname{sp}(\varphi) \subseteqq (-\infty, 0]\}$$

[1, p. 48], we find that $\operatorname{sp}(f\,d\mu) \subseteqq \{0\}$. By Proposition 2. 2, $f\,d\mu$ is an invariant measure; and since μ is ergodic by hypothesis, f is constant. Since, however, the measure $f\,d\mu$ annihilates the constants, $f = 0$. Whence $\mathfrak{A} + \overline{\mathfrak{A}}$ is weak-$*$ dense in $L^\infty(\mu)$ and the proof of Theorem I is complete.

If λ is in $M(X)$ and if t is in $\mathbf{R}$, we also write $T_t\lambda$ for the measure whose value at a Baire set E is $\lambda(E+t)$. The total variation measure of a measure λ in $M(X)$ will be denoted by $|\lambda|$. Recall that an arbitrary measure λ in $M(X)$ is called quasi-invariant in case $T_t|\lambda|$ and $|\lambda|$ have the same null sets for each t in $\mathbf{R}$.

The proof of the following lemma is a straightforward application of the definition of the term total variation measure and so will not be given.

Lemma 2. 3. *Let λ be a measure in $M(X)$ which is invariant under the action of $\mathbf{R}$ on X. Then $|\lambda|$ also is an invariant measure and the Radon—Nikodym derivative $\frac{d\lambda}{d|\lambda|}$ is (after modification on a $|\lambda|$-null set) an invariant function on X.*

Proof of Theorem II:

The unique probability measure on X which is invariant under the action of $\mathbf{R}$ will be denoted by μ. Recall that μ is necessarily ergodic.

We must show that $\mathfrak{A}+\overline{\mathfrak{A}}$ is uniformly dense in $C(X)$. Suppose the contrary and let λ be a measure in $M(X)$ of unit norm which annihilates $\mathfrak{A}+\overline{\mathfrak{A}}$. Then by Proposition 2′ of [1] and the fact that $\overline{\mathfrak{A}}=\{\varphi\in C(X)|\mathrm{sp}(\varphi)\subseteqq(-\infty,0]\}$ [1, p. 48], we see that $\mathrm{sp}(\lambda)\subseteqq\{0\}$. By Proposition 2. 2, λ is invariant and so, by Lemma 2. 3, $|\lambda|$ is invariant. Since λ has norm one, $|\lambda|$ is a probability measure, and by the strict ergodicity of $(X, \mathbf{R})$, $|\lambda|=\mu$. Since by Lemma 2. 3, $d\lambda/d|\lambda|$ is invariant and since $|\lambda|=\mu$ is ergodic, $d\lambda/d|\lambda|$ is constant. Thus λ is a constant multiple of μ. But the multiple must be zero because λ annihilates the constant functions, and thus we have arrived at a contradiction. Whence $\mathfrak{A}+\overline{\mathfrak{A}}$ is uniformly dense in $C(X)$ and the proof of Theorem II is complete.

§ 3. In this section we investigate the properties of representing measures for $\mathfrak{A}$ on X. We note in advance that we will use the following fact several times in our arguments. If m is an arbitrary representing measure for $\mathfrak{A}$ on X, then each real-valued function in $H^2(m)$ is constant (see [7, p. 98]).

For each t in $\mathbf{R}$ we let C_t denote the closure in $C(X)$ of the space of functions φ in $C(X)$ such that $\mathrm{sp}(\varphi)\subseteqq(t,\infty)$. By Lemma 3 of [1], C_t is a linear subspace of $C(X)$; and if $t\geqq 0$, C_t is an ideal in $\mathfrak{A}$ by Theorem 1 of [1].

If $\{M_\alpha\}_{\alpha\in A}$ is a family of closed subspaces of a Hilbert space then $\bigvee_{\alpha\in A} M_\alpha$ will denote their span and $\bigwedge_{\alpha\in A} M_\alpha$ will denote their intersection. Similarly, if $\{P_\alpha\}_{\alpha\in A}$ is a family of orthogonal projections on a Hilbert space, then $\bigvee_{\alpha\in A} P_\alpha$ will denote their least upper bound and $\bigwedge_{\alpha\in A} P_\alpha$ will denote their greatest lower bound.

If μ is a positive Baire measure on X, then we will often regard functions in $L^\infty(\mu)$ as multiplication operators on $L^2(\mu)$. Note that when $L^\infty(\mu)$ is regarded as an algebra of operators on $L^2(\mu)$, it is the closure of $C(X)$ in the weak operator topology; and observe that the subspaces of $L^2(\mu)$ which reduce $C(X)$ are of the form $\chi_E L^2(\mu)$ where χ_E denotes the characteristic function of the Baire set E.

Theorem III. *If m is a representing measure for $\mathfrak{A}$ on X which is not a point mass, then m is quasi-invariant under the action of* $\mathbf{R}$.

Proof. The proof rests on Theorem 2 of [1]. For each t in $\mathbf{R}$, let M_t be the closure of C_t in $L^2(m)$. The family $\{M_t\}_{t\in\mathbf{R}}$ is a decreasing family of subspaces of $L^2(m)$. Since the space of continuous functions on X with compact spectrum is dense in $C(X)$ [1, Lemma 3] and is contained in $\bigvee_{t\in\mathbf{R}} M_t$, it follows that $\bigvee_{t\in\mathbf{R}} M_t= =L^2(m)$. If φ is in $C(X)$ with $\mathrm{sp}(\varphi)\subseteqq(s,\infty)$, then $\varphi C_t\subseteqq C_{t+s}$ for all t by [1, Theorem 1] and so $\varphi M_t\subseteqq M_{t+s}$ for all t. From this it follows that $\bigwedge_{t\in\mathbf{R}} M_t$ is reduced by each continuous function with compact spectrum. Thus $\bigwedge_{t\in\mathbf{R}} M_t$ is reduced by $C(X)$, and, by the above remarks must be of the form $\chi_E L^2(m)$ for some Baire set E. Since $\chi_E L^2(m) = \bigwedge_{t\in\mathbf{R}} M_t \subseteqq H^2(m)$, the facts that $H^2(m)$ contains no non-constant real-valued functions and that m is not a point mass allow us to conclude that $\bigwedge_{t\in\mathbf{R}} M_t$ is the zero space. Thus we have shown that $\{M_t\}_{t\in\mathbf{R}}$ satisfies the hypotheses of part 3 in Theorem 2 of [1]. Whence, by that theorem, m is quasi-invariant, and the proof of Theorem III is complete.

Theorem IV. *If m is a representing measure for $\mathfrak{A}$ on X such that $H^2(m)$ contains functions other than constants, then m is ergodic if and only if no non-zero function in $H^2(m)$ vanishes on a set of positive measure.*

Proof. The hypothesis implies that m is not a point mass so that by Theorem III m is quasi-invariant.

Suppose m is ergodic and let f be a function in $H^2(m)$ which vanishes on a set of positive measure. If E is the support of f, then $m(E)<1$ and we must show that $m(E)=0$. Observe that the smallest subspace of $L^2(m)$ which contains f and reduces $C(X)$ is $\chi_E L^2(m)$. Let $d\mu=\chi_E dm$, identify $L^2(\mu)$ with $\chi_E L^2(m)$, and for each t in $\mathbf{R}$ let $\mathscr{K}_t$ be the closed linear span in $L^2(\mu)$ of the space $\{\varphi f | \varphi \in C_t\}$. It follows easily from the proof of Theorem III that $\bigvee_{t\in\mathbf{R}} \mathscr{K}_t = L^2(\mu)$ and that $\bigwedge_{t\in\mathbf{R}} \mathscr{K}_t = \chi_F L^2(\mu)$ for some Baire set F. Since $\bigwedge_{t\in\mathbf{R}} \mathscr{K}_t \subseteqq H^2(m)$, $m(F)=0$ as before; so $\bigwedge_{t\in\mathbf{R}} \mathscr{K}_t$ is the zero subspace of $L^2(\mu)$. Finally, since the family $\{\mathscr{K}_t\}_{t\in\mathbf{R}}$ is decreasing and since $\varphi\mathscr{K}_t \subseteqq \subseteqq \mathscr{K}_{t+s}$ for all φ in $C(X)$ with sp $(\varphi) \subseteqq (s, \infty)$ we may apply Theorem 2 of [1] again to conclude that μ is quasi-invariant. However, m is quasi-invariant and is also ergodic by hypothesis. Therefore, since μ is absolutely continuous with respect to m, μ must be the zero measure by Lemma 9 of [1]. Thus $m(E)=0$ as was to be shown.

To prove the converse, assume m is not ergodic and let E be an invariant Baire set such that $0<m(E)<1$. We will produce a non-zero function in $H^2(m)$ which is supported either on E or on $X-E$.

For each t in $\mathbf{R}$, let M_t be the closure of C_t in $L^2(m)$ and let P_t be the projection of $L^2(m)$ onto M_t. It was shown in the proof of Theorem III that since m is not a point mass on X, $\{M_t\}_{t\in\mathbf{R}}$ is a decreasing family of subspaces of $L^2(m)$ whose span is $L^2(m)$ and whose intersection is the zero subspace. It is also easy to see that for each t in $\mathbf{R}$, $M_t = \bigvee_{s>t} M_s$. Thus, except for orientation, the family $\{P_t\}_{t\in\mathbf{R}}$ is a resolution of the identity which is continuous from the right, i.e., $\{P_t\}_{t\in\mathbf{R}}$ has the following four properties: (i) $\bigvee_{t\in\mathbf{R}} P_t$ is the identity operator on $L^2(m)$; (ii) $\bigwedge_{t\in\mathbf{R}} P_t$ is the zero operator; (iii) if $t<s$, then $P_s \leqq P_t$; and (iv) for each t in $\mathbf{R}$, $P_t = \bigvee_{s>t} P_s$.

Let $\{V_t\}_{t\in\mathbf{R}}$ be the strongly continuous unitary representation of $\mathbf{R}$ on $L^2(m)$ defined by the formula

$$V_t=\int_{-\infty}^{\infty} e^{-its}\,dP_s,\quad t\in\mathbf{R}.$$

Then, as Forelli showed in the proof of Theorem 2 in [1],

$$V_t\varphi V_t^*=T_t\varphi \tag{3. 1}$$

for all φ in $C(X)$ and all t in $\mathbf{R}$. Since $L^\infty(m)$ is the closure of $C(X)$ in the weak operator topology, equation (3. 1) is valid for all functions in $L^\infty(m)$ provided, of course, that the right hand side of the equation is interpreted in the obvious way. Since E is invariant by assumption, equation (3. 1) implies that χ_E commutes with $\{V_t\}_{t\in\mathbf{R}}$. Whence χ_E commutes with $\{P_t\}_{t\in\mathbf{R}}$ and so leaves each M_t invariant. Because $H^2(m)$ is assumed to contain non-constant functions, it is not difficult to see that for some $t>0$, M_t contains non-zero functions. Let f be such a function and note that not both $\chi_E f$ and $\chi_{X-E} f$ can be zero. Since both these functions are in M_t and since $M_t\subseteqq H^2(m)$ for $t\geqq 0$, we see that $H^2(m)$ contains non-zero functions supported either on E or on $X-E$. Thus the proof of Theorem IV is complete.

A word of explanation concerning the hypothesis of Theorem IV is in order. If $\mathfrak{A}$ separates the points of X, then the hypothesis of Theorem IV follows from the hypothesis of Theorem III (see [7, p. 33]). However, examples show that $\mathfrak{A}$ need not always separate points; moreover, it is easy to see that on the occasions when this occurs there are representing measures m for $\mathfrak{A}$ such that m is not a point mass, m is not ergodic, and such that $H^2(m)$ contains only constant functions. We note that in numerous cases of particular interest $\mathfrak{A}$ separates points. For example, this happens when the flow is strictly ergodic and when it is minimal.

Observe that if m is a representing measure for $\mathfrak{A}$ on X such that $\mathfrak{A}$ is a weak-$*$ Dirichlet algebra in $L^\infty(m)$, then $H^2(m)$ consists solely of constants if and only if m is a point mass. Observe also that if $\mathfrak{A}$ is a weak-$*$ Dirichlet algebra in $L^\infty(m)$ then the space $H^2(m)$ has the property that no non-zero function in it vanishes in a set of positive measure if and only if each $H^p(m)$ has this property, $1\leqq p\leqq\infty$ (see [12]). Thus we find that the following corollary is a consequence of Theorem IV and [12].

Corollary 3. 1. *Let m be a representing measure for $\mathfrak{A}$ on X which is not a point mass and such that $\mathfrak{A}$ is a weak-$*$ Dirichlet algebra in $L^\infty(m)$. Then the following assertions are equivalent:*

(i) *m is ergodic;*

(ii) *for $1\leqq p\leqq\infty$, no non-zero function in $H^p(m)$ can vanish on a set of positive measure;*

(iii) *$H^\infty(m)$ is a maximal weak-$*$ closed subalgebra of $L^\infty(m)$.*

Our final goal is to show that when the flow is strictly ergodic each representing measure for $\mathfrak{A}$ which is not a point mass on X is ergodic. To achieve this goal we prove the following result which is interesting in its own right.

Theorem V. *Suppose that the flow $(X, \mathbf{R})$ is strictly ergodic, and let μ be the unique invariant probability measure on X. Then C_0 is a maximal ideal in $\mathfrak{A}$ and μ is its (necessarily unique) representing measure.*

Proof. The proof is very similar to part (i) in the proof of Theorem I. Let $\mathfrak{A}_\mu = \{\varphi \in \mathfrak{A} | \langle \varphi, \mu \rangle = 0\}$. Since μ is invariant the arguments in the proof of Theorem I show that $C_0 \subseteqq \mathfrak{A}_\mu$. Therefore, to complete the proof, it clearly suffices to show $\mathfrak{A}_\mu \subseteqq C_0$. To this end, suppose φ is in $\mathfrak{A}_\mu$ but not in C_0 and choose a measure λ in $M(X)$ which annihilates C_0 while satisfying the equation $\langle \varphi, \lambda \rangle = 1$. By Proposition 2 of [1], λ is analytic. Consequently, if $F(t) = \langle T_t \varphi, \lambda \rangle$ then as before F is the constant one. For positive τ we set $\varphi_\tau = \frac{1}{2r} \int_{-\tau}^{\tau} (T_t \varphi) dt$. Then for each such τ, $\langle \varphi_\tau, \lambda \rangle = \frac{1}{2\tau} \int_{-\tau}^{\tau} F(t) dt = 1$ and so

$$\lim_{\tau \to \infty} \langle \varphi_\tau, \lambda \rangle = 1.$$

In the proof of Theorem I we invoked the individual ergodic theorem to obtain a contradiction. However, that result has no bearing here. Instead we appeal to Théorème XV on page 107 of [11] and the strict ergodicity of the flow $(X, \mathbf{R})$ to conclude that the φ_τ converge pointwise (everywhere) to $\langle \varphi, \mu \rangle = 0$. Since the φ_τ are uniformly bounded we see that

$$\lim_{\tau \to \infty} \langle \varphi_\tau, \lambda \rangle = 0.$$

This contradiction completes the proof.

Theorem VI. *If the flow $(X, \mathbf{R})$ is strictly ergodic and if m is an arbitrary representing measure for $\mathfrak{A}$ on X which is not a point mass, then m is ergodic.*

Proof. Let μ be the unique invariant probability measure on X, recall that by Theorem V μ represents the maximal ideal C_0 in $\mathfrak{A}$, and consider the following two mutually exclusive and exhaustive cases.

Case 1. $\|\mu - m\| < 2$. In this case, μ and m represent points in the same Gleason part of the maximal ideal space of $\mathfrak{A}$ and hence are mutually absolutely continuous [7, p. 143]. Thus since μ is ergodic, so is m.

Case 2. $\|\mu - m\| = 2$. In this case, μ and m represent points in distinct Gleason parts of the maximal ideal space of $\mathfrak{A}$ and so μ and m are mutually singular [7, p. 144]. Since C_0 is the maximal ideal in $\mathfrak{A}$ represented by μ (Theorem V), the abstract Kolmogoroff—Krein Theorem [7, p. 135] implies that C_0 is dense in $H^2(m)$.

For each t in $\mathbf{R}$, let M_t be the closure of C_t in $L^2(m)$ and let P_t be the projection of $L^2(m)$ onto M_t. In the proof of Theorem IV it was shown that $\{P_t\}_{t\in\mathbf{R}}$ is a resolution of the identity whose Fourier—Stieltjes transform $\{V_t\}_{t\in\mathbf{R}}$ is a strongly continuous unitary representation of $\mathbf{R}$ on $L^2(m)$ which satisfies equation (3. 1). By equation (3. 1) we know that if E is any invariant Baire set, then χ_E leaves each M_t invariant. This is true in particular for M_0. However, the conclusion of the preceding paragraph is that $M_0=H^2(m)$. It follows from this and the fact that $H^2(m)$ contains no non-constant real-valued functions that if E is an invariant Baire set, then $m(E)$ is zero or one. Whence, in case 2 also, m is ergodic and the proof is complete.

§ 4. One problem which arises at this point is to determine the structure of the maximal ideal space of $\mathfrak{A}$. We have been able to show that in the strictly ergodic case a point in the maximal ideal space of $\mathfrak{A}$, which is not in $X\cup\{C_0\}$, lies in a non-trivial Gleason part. Moreover, C_0 lies in a non-trivial Gleason part if and only if the unique invariant probability measure is supported on a non-trivial orbit. On the basis of these facts and what is known for spaces of generalized analytic functions [7, p. 171], we conjecture that in the strictly ergodic case, at least, the maximal ideal space of $\mathfrak{A}$ is homeomorphic to $X\times[0, 1]$ with the slice $X\times\{0\}$ identified to a point.[1)]

It appears that virtually all of the Helson—Lowdenslager invariant subspace theory [10] is valid in our setting and that an analysis similar to that in [13] may be developed to determine the Hilbert space representations of the algebra of analytic functions associated with a flow. We hope to pursue these matters in a future article.

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1) *Added in proof:* This conjecture is correct, at least if X is separable and the unique invariant measure is not concentrated on a periodic orbit. For the proof, see our paper: Function Algebras and Flows. II, which will appear in *Arkiv för Matematik.*

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The Denjoy—Luzin theorem on trigonometric series

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Let $\{g_n\}$ be a sequence of transformations from the real line to the real line. Throughout this paper assume that f is periodic with period $p = b-a$ (>0), $f \in L^1[a, b]$, and set $f_n(x) = f(g_n(x))$. The sequence $\{g_n\}$ will be called an *A-sequence with respect to* f if the absolute convergence of $\Sigma c_n f_n(x)$ in a set of positive measure implies that $\Sigma |c_n|$ converges. The classical Denjoy—Luzin Theorem on trigonometric series states that $g_n(x) = nx + B_n$ is an A-sequence with respect to $f(x) = \cos x$, where $\{B_n\}$ is any sequence of numbers. The present author generalizes this result by considering the infimum of the averages of a function over sets of constant measure. The following result is a simple corollary. The sequence $g_n(x) = A_n x + B_n$ is an A-sequence for every periodic f that is essentially nonzero (i.e., nonzero almost everywhere) if and only if $\liminf |A_n| > 0$.

Theorem A (ORLICZ [1], p. 327). *The sequence $\{g_n\}$ is an A-sequence with respect to a periodic f if and only if*

$$\liminf_{n\to\infty} \int_E |f_n(x)|\, dx > 0$$

for every set E of positive measure.

Theorem B (FEJÉR [3], p. 48; [6], p. 49). *If $g_n(x) = A_n x + B_n$, where B_n are arbitrary real numbers and* $\lim |A_n| = \infty$, *then*

$$\lim_{n\to\infty} (\mu(E))^{-1} \int_E |f_n(x)|\, dx = (b-a)^{-1} \int_a^b |f(x)|\, dx$$

for every set E of positive measure.

Theorem 1. *The sequence $g_n(x) = A_n x + B_n$ is an A-sequence for every periodic f that is not essentially zero if and only if* $\lim |A_n| = \infty$.

Proof. *Sufficiency* (cf. [4], p. 84). Apply Theorems A and B.

Necessity. Let $f(x)=0$ on $(-1, 1)$, 1 on $(1, 3)$, and $f(x+4)=f(x)$. If $\liminf |A_n|<\infty$, then $\liminf |A_n \delta|<1$ for some $\delta>0$. Hence

$$\liminf_{n\to\infty} \int_0^\delta f(A_n x)\, dx=0.$$

Definition 1. If $h\in L^1[a, b]$, A is a measurable set, and $0<\delta\leqq\mu(A)$, then we define

$$m(h, \delta, A)=\inf_B\Big\{(\mu(B))^{-1}\int_B |h(t)|\, dt\Big\}$$

for measurable subsets B of A with $M(B)\geqq\delta$, and we define

$M(h, \delta, A)=$ supremum of the above set of averages.

Also if $A=[a, b]$, we simply write $m(h, \delta)$ or $M(h, \delta)$ for any function h under consideration.

Remark 1. From Definition 1, we obtain

$$M(h, \delta)\uparrow \|h\|_\infty\equiv \operatorname{ess\,sup}|h| \quad \text{as} \quad \delta\downarrow 0,$$
$$m(h, \delta)\downarrow \|h\|_0 \equiv \operatorname{ess\,inf}|h| \quad \text{as} \quad \delta\downarrow 0.$$

Lemma 1. *Let $h\in L^1[a, b]$. Then h is essentially nonzero if and only if for every measurable set A $m(f, \delta, A)>0$ whenever $0<\delta\leqq\mu(A)<\infty$.*

Proof. Assume h is essentially nonzero and $0<\delta\leqq\mu(A)<\infty$. Then for $B\subset A$ and $\mu(B)\geqq\delta$ we set

$$H_n=\{x\in B: |h(x)|>n^{-1}\}$$

and choose N such that $\mu(H_N)>\mu(B)/2$. Then, we obtain

$$\int_B |h(t)|\, dt\geqq\mu(H_N)/N\geqq\mu(B)/2N.$$

Lemma 2 ([5], p. 315; [2], p. 1245). *If g is a strictly monotonic absolutely continuous function on $[a, b]$ with range $[c, d]$ $(\{g(x): x\in[a, b]\}=[c, d])$, then for every measurable set $E\subset[a, b]$*

a)
$$\int_E f(g(x))|g'(x)|\, dx=\int_{g(E)} f(y)\, dy$$

and

b)
$$\delta m(g', \delta)\leqq\int_E |g'(x)|\, dx=\mu(gE)\leqq\delta M(g', \delta) \quad \textit{if} \quad \mu(E)=\delta.$$

From Definition 1 we also obtain

Lemma 3. *If n is a positive integer and $0<\delta\leqq np$, then*

$$m(f, \delta, [0, np])=m(f, \delta/n).$$

Definition 2. If e is a real number we let e^* and e_* denote respectively the least integer greater than or equal to e and the greatest integer less than or equal to e.

Theorem 2. *Let g be a strictly monotonic absolutely continuous function with domain $[a, b]$ and range $[c, d]$. If $e = \|g'\|_1/p = (d-c)/p$ and $f\circ g$ is the function $f(g(x))$, then*

$$\frac{m(g', \delta)}{\|g'\|_\infty} m\left(f, \frac{\delta m(g', \delta)}{e^*}\right) \leqq m(f\circ g, \delta) \leqq \frac{M(g', \delta)}{\|g'\|_0} m\left(f, \frac{\delta M(g', \delta)}{e_*}\right)$$

where for the left inequality we assume $\|g'\|_\infty$ is finite and $0<\delta\leqq p$, and for the right inequality we assume $\|g'\|_0$ and e_* are nonzero and $0<\delta M(g', \delta)/e_*\leqq p$.

Proof. By Lemmas 2 and 3

$$m(f\circ g, \delta)\|g'\|_\infty \geqq m((f\circ g)g', \delta) \geqq m(g', \delta)m(f, \delta m(g', \delta), [c, d]) \geqq$$

$$\geqq m(g', \delta)m(f, \delta m(g', \delta), [c, c+e^*p]) = m(g', \delta)m(f, \delta m(g', \delta)/e^*).$$

The proof of the right half is similar.

Corollary 1. *If $\{g_n\}$ is a sequence of strictly monotonic absolutely continuous functions with domain $[a, b]$, and also for each g_n and each δ such that $0<\delta\leqq p$, we have*

$$(\|g_n'\|_\infty)^*\leqq K(\delta)m(g_n', \delta)$$

where $K(\delta)$ is a positive constant dependent only on δ, then $\{g_n\}$ is an A-sequence for every periodic f that is essentially nonzero.

Corollary 2. *If $A\neq 0$, $g(x) = Ax+B$, and we set $e=|A|$, then*

$$m(f, \delta e/e^*) \leqq m(f\circ g, \delta) \leqq m(f, \delta e/e_*) \tag{1}$$

where for the left inequality we assume $0<\delta\leqq p$, and for the right inequality we assume $e\geqq 1$ and $0<\delta e/e_*\leqq p$.

Corollary 3. *The sequence $g_n(x) = A_n x+B_n$ is an A-sequence for every periodic f that is essentially nonzero if and only if* $\liminf |A_n|>0$.

Proof. *Sufficiency.* If $\liminf A_n>0$, then $\liminf A_n/A_n^*>0$. Now, our result follows from Theorem A, Lemma 1, and Corollary 2.

Necessity. If $f(x)=x$ on $[0, 1]$, then

$$\liminf_{n\to\infty}\int_0^\delta |f(A_n x)|\,dx=(\delta^2/2)\liminf_{n\to\infty}|A_n|$$

and so our result follows by Theorem A.

Remark 2. The use of e^* and e_* is necessary in Theorem 2 and its corollaries. In fact, Corollary 2 is best possible in a certain sense. For example, if $f(x)=x$ on $[0, 1]=[a, b]$ and $g(x)=ex$ for a noninteger $e>0$, then equality is attained in the left side of (1) for $\delta < e^*(1-e_*/e)$. Similarly, if $f(x) = 1-x$ on $[0, 1]=[a, b]$ and $g(x)=ex$ for a noninteger $e>1$, then equality is attained in the right side of (1) for $\delta < e_*(e^*/e-1)$.

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Generalizations of the Hardy—Littlewood inequality. II

By JÓZSEF NÉMETH in Szeged

1. G. H. HARDY and J. E. LITTLEWOOD [2] proved the following

Theorem A. *Suppose that $a_n \geqq 0$ $(n=1, 2, \ldots)$ and that c is a real number. Set*

$$A_{m,n} = \sum_{\nu=m}^{n} a_\nu .$$

If $p>1$ we have

$$\sum_{n=1}^{\infty} n^{-c} A_{1,n}^{p} \leqq K \sum_{n=1}^{\infty} n^{-c} (n \cdot a_n)^p \quad \textit{with} \quad c>1, ^{*)} \tag{1}$$

$$\sum_{n=1}^{\infty} n^{-c} A_{n,\infty}^{p} \leqq K \sum_{n=1}^{\infty} n^{-c} (n \cdot a_n)^p \quad \textit{with} \quad c<1; \tag{2}$$

and if $0<p<1$ we have

$$\sum_{n=1}^{\infty} n^{-c} A_{1,n}^{p} \geqq K \sum_{n=1}^{\infty} n^{-c} (n \cdot a_n)^p \quad \textit{with} \quad c>1, \tag{3}$$

$$\sum_{n=1}^{\infty} n^{-c} A_{n,\infty}^{p} \geqq K \sum_{n=1}^{\infty} n^{-c} (n \cdot a_n)^p \quad \textit{with} \quad c<1. \tag{4}$$

The inequalities (1) and (2) were generalized by H. P. MULHOLLAND [4], moreover (3) and (4) by CHEN YUNG MING [1], replacing the function x^p in (1)—(4) by more general functions, notably they proved inequalities of the following type

$$\sum_{n=1}^{\infty} n^{-c} \Phi(A_{1,n}) \leqq K \sum_{n=1}^{\infty} n^{-c} \Phi(n a_n), \tag{5}$$

$$\sum_{n=1}^{\infty} n^{-c} \Psi(A_{1,n}) \geqq K \sum_{n=1}^{\infty} n^{-c} \Psi(n a_n) \tag{6}$$

under certain conditions on the functions $\Phi(x)$, $\Psi(x)$ and C.

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*) K denotes a positive absolute constant, not necessarily the same at each occurrence.

Theorem A was generalized in another direction by L. Leindler [3], who replaced in (1)—(4) the sequence $\{n^{-c}\}$ by an arbitrary sequence $\{\lambda_n\}$; for instance he proved the following inequality:

$$\sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \leqq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \left(\sum_{m=n}^{\infty} \lambda_m \right)^p a_n^p \tag{7}$$

with $p \geqq 1$ and $\lambda_n > 0$.

In the present paper we prove a theorem which contains all of these results.

2. We use the following notations:

a) $\Phi(x)$ $(x \geqq 0)$ denotes a non-negative function such that $\varphi(x) = \Phi(x)/x$ is increasing and, for some $k > 1$, $f(x) = \Phi(x)/x^k$ is decreasing.

b) $\Psi(x)$ $(x \geqq 0)$ denotes a non-negative function increasing to infinity such that $\varrho(x) = \Psi(x)/x$ is decreasing to zero, when x is increasing from zero to infinity.

c) $\Lambda_{m,n} = \sum_{i=m}^{n} \lambda_i \quad (1 \leqq m \leqq n \leqq \infty)$.

3. We prove the following

Theorem. *If $a_n \geqq 0$ and $\lambda_n > 0$ $(n = 1, 2, \ldots)$, then*

$$\sum_{n=1}^{\infty} \lambda_n \Phi(\Lambda_{1,n}) \leqq K_1 \sum_{n=1}^{\infty} \lambda_n \Phi\left(\frac{a_n}{\lambda_n} \Lambda_{n,\infty} \right), \tag{8}$$

and

$$\sum_{n=1}^{\infty} \lambda_n \Phi(\Lambda_{n,\infty}) \leqq K_2 \sum_{n=1}^{\infty} \lambda_n \Phi\left(\frac{a_n}{\lambda_n} \Lambda_{1,n} \right), \tag{9}$$

where K_1 and K_2 are constants depending on Φ; furthermore

$$\sum_{n=1}^{\infty} \lambda_n \Psi\left(\frac{a_n}{\lambda_n} \Lambda_{n,\infty} \right) \leqq C_1 \sum_{n=1}^{\infty} \lambda_n \Psi(\Lambda_{1,n}) \tag{10}$$

and

$$\sum_{n=1}^{\infty} \lambda_n \Psi\left(\frac{a_n}{\lambda_n} \Lambda_{1,n} \right) \leqq C_2 \sum_{n=1}^{\infty} \lambda_n \Psi(\Lambda_{n,\infty}), \tag{11}$$

where C_1 and C_2 are positive absolute constants.

4. We remark that this theorem implies Leindler's theorem [3] and several results of Chen Yung Ming [1] and H. P. Mulholland [4]; the method of proof of (10) and (11) is similar to that of Leindler's theorem.

5. We require the following lemmas:

Lemma 1. *If $\Phi(x)$ and $\varphi(x)$ are the functions defined above and $a_\nu \geqq 0$, then*

$$\Phi(\Lambda_{1,n}) \leqq K \sum_{\nu=1}^{n} a_\nu \varphi(\Lambda_{1,\nu}).$$

Lemma 2. *If $\Phi(x)$ and $\varphi(x)$ are the functions defined above, and $a_\nu \geqq 0$, then for every natural number N*

$$\Phi(A_{n,N}) \leqq K \sum_{\nu=n}^{N} a_\nu \varphi(A_{\nu,N}).$$

Lemma 3. *If $b_n > 0$, $c_n \geqq 0$, $a_n \geqq 0$ $(n=1, 2, \ldots)$ and if for every natural number N*

$$\sum_{n=1}^{N} b_n \Phi(A_{n,N}) \leqq K \sum_{n=1}^{N} c_n,$$

then

$$\sum_{n=1}^{\infty} b_n \Phi(A_{n,\infty}) \leqq K \sum_{n=1}^{\infty} c_n.$$

6. Proof of Lemma 1. Let $f(x)$ be the function defined above, in point 2, and write A_n instead of $A_{1,n}$. Then

$$\Phi(A_n) = \Phi(A_1) + \Phi(A_2) - \Phi(A_1) + \cdots + \Phi(A_n) - \Phi(A_{n-1});$$

as

$$\Phi(A_m) - \Phi(A_{m-1}) = f(A_m) A_m^k - f(A_{m-1}) A_{m-1}^k \leqq$$

$$\leqq k f(A_m) A_m^{k-1} (A_m - A_{m-1}) = k\varphi(A_m) a_m \quad \text{for} \quad m \geqq 2,$$

and $\Phi(A_1) \leqq k\varphi(A_1) a_1$, we obtain the assertion.

Proof of Lemma 2. Let us write B_n for $A_{n,N}$ $(N \geqq n)$. Then

$$\Phi(B_n) = \Phi(B_n) - \Phi(B_{n+1}) + \cdots + \Phi(B_{N-1}) - \Phi(B_N) + \Phi(B_N);$$

using the estimations

$$\Phi(B_m) - \Phi(B_{m+1}) = f(B_m) B_m^k - f(B_{m+1}) B_{m+1}^k \leqq k\varphi(B_m)(B_m - B_{m+1}) = k\varphi(B_m) a_m,$$

and

$$\Phi(B_N) \leqq k\varphi(B_N) a_N,$$

we obtain the assertion.

Proof of Lemma 3. This can be done by an easy computation.

7. Proof of the Theorem. *Inequality* (8). Applying Lemma 1 we obtain that

$$\sum_{n=1}^{N} \lambda_n \Phi(A_{1,n}) \leqq k \sum_{n=1}^{N} \lambda_n \sum_{\nu=1}^{n} a_\nu \varphi(A_{1,\nu}) = \sum\nolimits_1$$

holds for every natural number N.

Interchanging the order of the summations we have

$$\sum_1 = k\sum_{\nu=1}^{N} a_\nu \varphi(A_{1,\nu})\Lambda_{\nu,N} = k\sum_{\nu=1}^{N} t^{-1}\left\{t\frac{a_\nu}{\lambda_\nu}\Lambda_{\nu,N}\varphi(A_{1,\nu})\lambda_\nu\right\}.$$

Since

$$x\varphi(y) \leqq x\varphi(x)+y\varphi(y) = \Phi(x)+\Phi(y) \quad \text{for } x\geqq 0,\ y\geqq 0 \tag{12}$$

and

$$\Phi(tx) = f(tx)t^k x^k \leqq t^k f(x)x^k = t^k\Phi(x) \quad \text{for } t\geqq 1,\ x\geqq 0, \tag{13}$$

we obtain

$$\sum_1 \leqq k\sum_{\nu=1}^{N} t^{-1}\left\{\Phi\left(t\frac{a_\nu}{\lambda_\nu}\Lambda_{\nu,N}\right)\lambda_\nu + \Phi(A_{1,\nu})\lambda_\nu\right\} \leqq \tag{14}$$

$$\leqq k\sum_{\nu=1}^{N}\left\{t^{k-1}\Phi\left(\frac{a_\nu}{\lambda_\nu}\Lambda_{\nu,N}\right)\lambda_\nu + t^{-1}\Phi(A_{1,\nu})\lambda_\nu\right\}.$$

Hence, choosing t such that $1-kt^{-1}$ be positive, we obtain

$$\sum_{n=1}^{N}\lambda_n\Phi(A_{1,n}) \leqq \frac{k\cdot t^{k-1}}{1-k\cdot t^{-1}}\sum_{n=1}^{N}\lambda_n\Phi\left(\frac{a_n}{\lambda_n}\Lambda_{n,N}\right),$$

which proves (8).

Inequality (9). Applying Lemma 2 we have, for an arbitrary natural number N,

$$\sum_{n=1}^{N}\lambda_n\Phi(A_{n,N}) \leqq k\sum_{n=1}^{N}\lambda_n\sum_{\nu=n}^{N}a_\nu\varphi(A_{\nu,N}) = \sum_2.$$

Interchanging the order of the summations we obtain

$$\sum_2 = k\sum_{\nu=1}^{N}a_\nu\varphi(A_{\nu,N})\Lambda_{1,n} = k\sum_{\nu=1}^{N}t^{-1}\left\{t\frac{a_\nu}{\lambda_\nu}\Lambda_{1,n}\varphi(A_{\nu,N})\lambda_\nu\right\}.$$

Using (12) and (13), similarly to (14), we have

$$\sum_2 \leqq k\sum_{\nu=1}^{N}\left\{t^{k-1}\Phi\left(\frac{a_\nu}{\lambda_\nu}\Lambda_{1,\nu}\right)\lambda_\nu + t^{-1}\Phi(A_{\nu,N})\lambda_\nu\right\}.$$

Hence, if $1-kt^{-1} > 0$ we get

$$\sum_{n=1}^{N}\lambda_n\Phi(A_{n,N}) \leqq \frac{kt^{k-1}}{1-kt^{-1}}\sum_{n=1}^{N}\lambda_n\Phi\left(\frac{a_n}{\lambda_n}\Lambda_{1,n}\right),$$

and using Lemma 3, we obtain (9).

Inequality (10). We may suppose that the series on the right-hand side is convergent, and thus we can define a sequence $\{m_n\}$ in the following way:

Let $m_0=0$ and for $n\geqq 1$ let m_n be the smallest natural number $k(>m_{n-1})$ such that $\Lambda_{m_{n-1}+1,k}\geqq \Lambda_{k+1,\infty}$. Then $\Lambda_{m_n+1,m_{n+1}}\geqq \Lambda_{m_{n+1}+1,\infty}$, and

$$\Lambda_{m_n+1,m_{n+1}}\leqq 2\Lambda_{m_{n+1},m_{n+2}}, \tag{15}$$

and

$$\Lambda_{m_n+1,\infty}\leqq 2\Lambda_{m_n+1,m_{n+1}}. \tag{16}$$

We first show that

$$\sum_{\nu=m_n+1}^{m_{n+1}}\lambda_\nu\Psi\left(\Lambda_{m_n+1,\infty}\cdot\frac{a_\nu}{\lambda_\nu}\right)\leqq 2\Psi(\Lambda_{m_n+1,m_{n+1}})\Lambda_{m_n+1,\infty}. \tag{17}$$

We use the following notations:

$$\tau_\nu^{(n)}=\frac{\Lambda_{m_n+1,m_{n+1}}}{\lambda_\nu}$$

Using the properties of the functions $\Psi(x)$, $\varrho(x)$ we get:

$$\sum_{\nu=m_n+1}^{m_{n+1}}\lambda_\nu\Psi\left(\Lambda_{m_n+1,\infty}\cdot\frac{a_\nu}{\lambda_\nu}\right)\leqq \Lambda_{m_n+1,\infty}\sum_{\nu=m_n+1}^{m_n+1}a_\nu\varrho(\tau_\nu^{(n)}a_\nu). \tag{18}$$

Let ν_i, $\bar\nu_j$ denote the subscripts such that $m_n+1\leqq \nu_i,\ \bar\nu_j\leqq m_{n+1}$, and

$$\tau_{\nu_i}^{(n)}a_{\nu_i}\leqq A_{m_n+1,m_{n+1}},\qquad \tau_{\bar\nu_j}^{(n)}a_{\bar\nu_j}>A_{m_n+1,m_{n+1}}.$$

Then

$$\sum_{\nu=m_n+1}^{m_{n+1}}a_\nu\varrho(\tau_\nu^{(n)}a_\nu)=\sum_{\nu=m_n+1}^{m_{n+1}}\frac{a_\nu\tau_\nu^{(n)}\varrho(\tau_\nu^{(n)}a_\nu)}{\tau_\nu^{(n)}}\leqq\sum_i^{(1)}\frac{A_{m_n+1,m_{n+1}}\varrho(A_{m_n+1,m_{n+1}})}{\tau_{\nu_i}^{(n)}}+$$

$$+\sum_j^{(2)}\varrho(A_{m_n+1,m_{n+1}})a_{\bar\nu_j}\leqq A_{m_n+1,m_{n+1}}\varrho(A_{m_n+1,m_{n+1}})\sum_i^{(1)}\frac{1}{\tau_{\nu_i}^{(n)}}+$$

$$+\varrho(A_{m_n+1,m_{n+1}})\sum_j^{(2)}a_{\bar\nu_j}\leqq 2\Psi(A_{m_n+1,m_{n+1}}).$$

thus, by (18), we get (17). Using (15), (16) and (17) we have:

$$\sum_{\nu=m_n+1}^{m_{n+1}}\lambda_\nu\Psi\left(\Lambda_{\nu,\infty}\frac{a_\nu}{\lambda_\nu}\right)\leqq 2\Lambda_{m_n+1,\infty}\Psi(A_{m_n+1,m_{n+1}})\leqq$$

$$\leqq 2\Lambda_{m_n+1,\infty}\Psi(A_{1,m_{n+1}})\leqq 4\Lambda_{m_n+1,m_{n+1}}\Psi(A_{1,m_{n+1}})\leqq 8\sum_{k=m_{n+1}}^{m_{n+2}}\lambda_k\Psi\left(\sum_{\nu=1}^{k}a_\nu\right).$$

Hence

$$\sum_{\nu=1}^{\infty} \lambda_\nu \Psi\left(\Lambda_{\nu,\infty} \frac{a_\nu}{\lambda_\nu}\right) \leqq 8 \sum_{n=0}^{\infty} \sum_{\nu=m_{n+1}}^{m_{n+2}} \lambda_\nu \Psi\left(\sum_{k=1}^{\nu} a_k\right) \leqq 16 \sum_{\nu=1}^{\infty} \lambda_\nu \Psi\left(\sum_{k=1}^{\nu} a_k\right),$$

which gives (10).

Inequality (11). We distinguish two cases. First we suppose

$$\Lambda_{1,\infty} < \infty.$$

We define a sequence of integers $\mu_0, \mu_1, \dots$. We set $\mu_0=0$, $\mu_1=1$ and if μ_n has already been defined we choose $\mu'_{n+1}=k$, where $k(>\mu_n)$ denotes the smallest integer satisfying

$$\Lambda_{\mu_n+1,k} \geqq 3\Lambda_{\mu_{n-1}+1,\mu_n} \tag{19}$$

provided such a k exists. If $\mu'_{n+1} > \mu_n+1$ then let $\mu_{n+1} = \mu'_{n+1}-1$ and if $\mu'_{n+1} = \mu_n+1$ then let $\mu_{n+1} = \mu_n+1$. If there exists no natural number k with (19) then let $\mu_{n+1}=\infty$. It is clear that this inductive definition always stops at some $n=N_0$, that is $\mu_{N_0}=\infty$ holds. For in opposite case, by the definition of μ_n, the inequality

$$3I_{n-2} \leqq I_{n-1}+I_n \tag{20}$$

holds for all $2 \leqq n < N_0$, where $I_n = \Lambda_{\mu_n+1,\mu_{n+1}}$ and inequality (20) for infinitely many n would imply $\Sigma\lambda_k=\infty$ contrary to the assumption. By (20) we have for $1 \leqq n < N_0-1$

$$\Lambda_{1,\mu_n} \leqq 3I_{n-1}+I_n \tag{21}$$

and

$$\Lambda_{1,\mu_{N_0}} \leqq 3I_{N_0-3}+5I_{N_0-2}. \tag{22}$$

Next we remark that

$$\sum_{\nu=\mu_n+1}^{\mu_{n+1}} \lambda_\nu \Psi\left(\Lambda_{1,\nu} \frac{a_\nu}{\lambda_\nu}\right) \leqq 2\Lambda_{1,\mu_{n+1}} \Psi(A_{\mu_n+1,\mu_{n+1}}). \tag{23}$$

By the properties of the functions $\Psi(x)$, $\varrho(x)$ we have

$$\sum_{\nu=\mu_n+1}^{\mu_{n+1}} \lambda_\nu \Psi\left(\Lambda_{1,\nu} \frac{a_\nu}{\lambda_\nu}\right) \leqq \sum_{\nu=\mu_n+1}^{\mu_{n+1}} \lambda_\nu \Psi\left(\Lambda_{1,\mu_{n+1}} \frac{a_\nu}{\lambda_\nu}\right) =$$

$$= \Lambda_{1,\mu_{n+1}} \sum_{\nu=\mu_n+1}^{\mu_{n+1}} a_\nu \varrho\left(\Lambda_{1,\mu_{n+1}} \frac{a_\nu}{\lambda_\nu}\right) \leqq \Lambda_{1,\mu_{n+1}} \sum_{\nu=\mu_n+1}^{\mu_{n+1}} a_\nu \varrho\left(\Lambda_{\mu_n+1,\mu_{n+1}} \frac{a_\nu}{\lambda_\nu}\right).$$

Hence, applying the idea of proof of (17), we obtain (23) immediately.

Using (23) we get

$$\sum_{n=0}^{N_0-1}\sum_{\nu=\mu_n+1}^{\mu_{n+1}}\lambda_\nu\Psi\left(\Lambda_{1,\nu}\frac{a_\nu}{\lambda_\nu}\right)\leqq 2\sum_{n=0}^{N_0-1}\Lambda_{1,\mu_{n+1}}\Psi(A_{\mu_n+1,\mu_{n+1}})=\sum\nolimits_3. \tag{24}$$

By the definition of the sequence $\{\mu_n\}$ and by (21) we have

$$\begin{aligned}\sum\nolimits_3\leqq 2\sum_{n=0}^{1}\Lambda_{1,\mu_{n+1}}\Psi(A_{\mu_n+1,\mu_{n+1}})+2\sum_{n=2}^{N_0-2}\Lambda_{1,\mu_{n+1}}\Psi(A_{\mu_n+1,\mu_{n+1}})+\\ +2\Lambda_{1,\mu_{N_0}}\Psi(A_{\mu_{N_0-1}+1,\infty}) = \sum\nolimits_4+\sum\nolimits_5+\sum\nolimits_6.\end{aligned} \tag{25}$$

Using (19) and (21) we get

$$\begin{aligned}\sum\nolimits_5\leqq 2\sum_{n=2}^{N_0-2}(\Lambda_{1,\mu_{n-1}}+\Lambda_{\mu_{n-1}+1,\mu_{n+1}})\Psi(A_{\mu_n+1,\mu_{n+1}})\leqq\\ \leqq 2\sum_{n=2}^{N_0-2}(3\Lambda_{\mu_{n-2}+1,\mu_{n-1}}+2\Lambda_{\mu_{n-1}+1,\mu_n}+\Lambda_{\mu_n+1,\mu_{n+1}})\Psi(A_{\mu_n+1,\mu_{n+1}})\leqq\\ \leqq 2\sum_{n=2}^{N_0-2}(3\Lambda_{\mu_{n-2}+1,\mu_{n-1}}+5\Lambda_{\mu_{n-1}+1,\mu_n}+\lambda_{\mu_n+1})\Psi(A_{\mu_n+1,\mu_{n+1}}).\end{aligned} \tag{26}$$

An easy computation gives by (19) and (21) that

$$\sum\nolimits_4 \leqq 2[5\lambda_1\Psi(A_{1,\infty})+\lambda_2\Psi(A_{2,\infty})]. \tag{27}$$

By (22) we obtain

$$\Lambda_{1,\mu_{N_0}}\Psi(A_{\mu_{N_0-1}+1,\infty})\leqq(3\Lambda_{\mu_{N_0-3}+1,\mu_{N_0-2}}+5\Lambda_{\mu_{N_0-2}+1,\mu_{N_0-1}})\Psi(A_{\mu_{N_0-1}+1,\infty}). \tag{28}$$

Using (26), (27), (28) we get

$$\sum\nolimits_4+\sum\nolimits_5+\sum\nolimits_6 \leqq 18\sum_{n=1}^{\infty}\lambda_n\Psi\left(\sum_{k=n}^{\infty}a_k\right),$$

which by (24) and (25) gives (11) in case $\Sigma\lambda_k<\infty$.

If $\Sigma\lambda_n=\infty$ then we define another index-sequence $\{m_n\}$. Let $m_0=0$ and $m_1=1$. If $m_0<m_1<\dots<m_n$ $(n\geqq 1)$ have been defined, then let m_{n+1} be the smallest natural number k with

$$\Lambda_{m_n+1,k}\geqq 2\Lambda_{m_{n-1}+1,m_n} \tag{29}$$

By the definition of m_n we have

$$\Lambda_{1,m_{n+1}-1} \leqq \Lambda_{m_{n+1},m_{n+1}-1}+2\Lambda_{m_{n-1},m_n}, \tag{30}$$

$$\Lambda_{1,m_2-1}\leqq 3\lambda_1, \tag{31}$$

$$\Lambda_{m_{n-1},m_{n+1}-1}\leqq 3\Lambda_{m_{n-1},m_n}. \tag{32}$$

First we remark that similarly to the proof of (17) and (23) we obtain the following inequalities

$$\sum_{k=m_n}^{m_{n+1}-1} \lambda_k \cdot \Psi\left(A_{m_{n-1}, m_{n+1}-1} \frac{a_k}{\lambda_k}\right) \leqq 2A_{m_{n-1}, m_{n+1}-1} \Psi(A_{m_n, m_{n+1}-1}), \tag{33}$$

$$\sum_{k=1}^{m_2-1} \lambda_k \Psi\left(A_{1, m_2-1} \frac{a_k}{\lambda_k}\right) \leqq 2A_{1, m_2-1} \Psi(A_{1, m_2-1}). \tag{34}$$

By the definition of sequence $\{m_n\}$ and by (30) we have

$$\sum_{k=1}^{\infty} \lambda_k \Psi\left(A_{1,k} \frac{a_k}{\lambda_k}\right) \leqq \sum_{n=1}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} \lambda_k \Psi\left(A_{1, m_{n+1}-1} \frac{a_k}{\lambda_k}\right) \leqq$$

$$\leqq \sum_{k=1}^{m_2-1} \lambda_k \Psi\left(A_{1, m_2-1} \frac{a_k}{\lambda_k}\right) + \sum_{n=2}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} \lambda_k \Psi\left[\frac{a_k}{\lambda_k}(A_{m_n+1, m_{n+1}-1} + 2A_{m_{n-1}, m_n})\right] = \sum\nolimits_7 .$$

Since $\Psi(2x) \leqq 2\Psi(x)$,

$$\sum\nolimits_7 \leqq \sum_{k=1}^{m_2-1} \lambda_k \Psi\left(A_{1, m_2-1} \frac{a_k}{\lambda_k}\right) + 2 \sum_{n=2}^{\infty} \sum_{k=m_n}^{m_{n+1}-1} \lambda_k \Psi\left(A_{m_{n-1}, m_{n+1}-1} \frac{a_k}{\lambda_k}\right) = \sum\nolimits_8 .$$

By (33) and (34),

$$\sum\nolimits_8 \leqq 2A_{1, m_2-1} \Psi(A_{1, m_2-1}) + 4 \sum_{n=2}^{\infty} A_{m_{n-1}, m_{n+1}-1} \Psi(A_{m_n, m_{n+1}-1}) = \sum\nolimits_9 .$$

Using (31), (32) we get

$$\sum\nolimits_9 \leqq 6\lambda_1 \Psi(A_{1,\infty}) + 12 \sum_{n=2}^{\infty} A_{m_{n-1}, m_n} \Psi(A_{m_n, \infty}) \leqq 24 \sum_{n=1}^{\infty} \lambda_n \Psi\left(\sum_{\nu=n}^{\infty} a_\nu\right),$$

which is the required inequality (11). The proof is complete.

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The centroid of a semigroup

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1. Introduction and summary. The centroid of a ring R is defined as the centralizer in the ring of all endomorphisms of the additive group of R of the ring generated by all left and right multiplications [10]. It reduces essentially to the center of R if R has an identity element. We borrow the multiplicative part of the definition of a centroid and apply this notion to the theory of semigroups. Hence the centroid $Z(S)$ of a semigroup S is the semigroup under composition of all transformations on S which are simultaneously left and right translations of S (written, say, as left operators). We exploit this concept for two principal purposes. The first one is connected with a generalization of inner automorphisms and the second one is a consideration of the congruence on the semigroup whose classes are the orbits of the group of units of the centroid. We study this congruence in detail for several classes of semigroups, but the most fruitful classes turn out to be cancellative and commutative cancellative semigroups. Based on this congruence, certain semigroups are isomorphic to a Schreier extension of an abelian group and another semigroup and can be embedded into a wreath product of these.

Section 2 is a preliminary one and contains most of the background needed throughout the paper. In Section 3 we introduce a generalization of inner automorphisms, and for a wide class of semigroups prove a theorem which in the case of groups reduces to the familiar relationship between inner automorphisms and the center. In Section 4 we study the congruence σ_S on a semigroup S whose classes are the orbits of the group of units of the centroid $GZ(S)$ of S. We prove that a semigroups in which $GZ(S)$ acts simply transitively on each σ_S-class is isomorphic to a Schreier extension of $GZ(S)$ by S/σ_S. We also establish an isomorphism theorem for this representation of S. For such a Schreier extension, we prove in Section 5 that for a wide class of semigroups, the extension can be embedded (or densely embedded) in the wreath product of some related semigroups. For cancellative semigroups S, in Section 6 we find an expression for σ_S in terms of elements of S, and for every element of S, find a copy of $Z(S)$ defined on a subset of S with a new multiplication. We also consider an example exhibiting interesting features in this context. Finally, in Section 7 we deal with commutative cancellative semigroups.

Every such semigroup is isomorphic to a Schreier extension of an abelian group and a commutative cancellative semigroup Q in which $aQ=bQ$ always implies $a=b$, and conversely. We further extablish several properties of this decomposition of S and of the congruence σ_S. For the case of the additive semigroups of all nonnegative or of all positive integers, we compute all functions figuring in the Schreier extension. It should be noted already that what we call a Schreier extension is close but not identical with the concept of the Schreier product used in [12]. Furthermore, since we often write functions on the left, we apply the "left version" of the wreath product and use the notation "wl" instead of "wr".

2. Preliminaries. We begin by recalling the concepts needed throughout the paper; for undefined terms and notation the reader is referred to [2]. Let S be a semigroup and let x, y be arbitrary elements of S. A function λ on S written on the left is a *left translation* of S if $\lambda(xy)=(\lambda x)y$; a function ϱ on S written on the right is a *right translation* if $(xy)\varrho=x(y\varrho)$; λ and ϱ are *linked* and we say that (λ, ϱ) is a *bitranslation* of S if $x(\lambda y)=(x\varrho)y$; λ and ϱ are *permutable* if $(\lambda x)\varrho=\lambda(x\varrho)$. The set $\Lambda(S)$ of all left translations of S under the composition $(\lambda\lambda')x=\lambda(\lambda' x)$ is a semigroup; the set $P(S)$ of all right translations is a semigroup under the composition $x(\varrho\varrho')=(x\varrho)\varrho'$. The set $\Omega(S)$ of all bitranslations of S with the multiplication induced by the direct product $\Lambda(S)\times P(S)$ is a semigroup called the *translational hull* of S. For $a\in S$, the functions λ_a and ϱ_a defined by $\lambda_a x=ax$ $x\varrho_a=xa$, are, respectively, the *inner left translation* and the *inner right translation* of S induced by a; $\pi_a=(\lambda_a, \varrho_a)$ is the *inner bitranslation* of S induced by a; $\Pi(S)=\{\pi_a|a\in S\}$ is the *inner part* of $\Omega(S)$.

It is easy to verify that

$$(1) \qquad (\lambda, \varrho)\pi_a=\pi_{\lambda a}, \quad \pi_a(\lambda, \varrho)=\pi_{a\varrho} \qquad (a\in S, (\lambda, \varrho)\in\Omega(S)),$$

which implies that $\Pi(S)$ is an ideal of $\Omega(S)$; one verifies similarly that $\Gamma(S)=\{\lambda_a|a\in S\}$ is a left ideal of $\Lambda(S)$ and that $\Delta(S)=\{\varrho_a|a\in S\}$ is a right ideal of $P(S)$. The projections

$$\pi_\Lambda:(\lambda, \varrho)\to\lambda, \quad \pi_P:(\lambda, \varrho)\to\varrho \qquad ((\lambda, \varrho)\in\Omega(S))$$

are homomorphisms, let

$$\tilde{\Lambda}(S)=\pi_\Lambda\Omega(S), \quad \tilde{P}(S)=\pi_P\Omega(S).$$

By $C(S)$ we denote the *center* of S, and if S has an identity, $G(S)$ denotes the *group of units* (invertible elements of S). As a generalization of the center of a semigroup, we borrow the following concept from the theory of rings: the *centroid* of S, denoted by $Z(S)$, is the set of all functions ζ on S satisfying $\zeta(xy)=(\zeta x)y=x(\zeta y)$ $(x, y\in S)$. It follows immediately that $Z(S)$ is a subsemigroup of $\Lambda(S)$

and is the centralizer of the set of all inner left and inner right translations of S if both of these are written as left operators (definition in [10], V, § 4).

If several operators are applied to S, we will retain the parentheses only around S, e.g., we write $C\Omega(S)$ instead of $C(\Omega(S))$, etc. If S has an identity e, then for any $\lambda \in \Lambda(S)$ and $\varrho \in P(S)$, we have $\lambda = \lambda_{\lambda e}$ and $\varrho = \varrho_{e\varrho}$, so $\Gamma(S) = \Lambda(S)$, $\Delta(S) = P(S)$, $\Omega(S) = \Pi(S)$, and $Z(S)$ can be identified with $C(S)$.

The transformation ι_S written on the left is the identity both of $Z(S)$ and $\Lambda(S)$; the pair (ι_S, ι_S), where the second ι_S is written on the right, is the identity of $\Omega(S)$. The groups $GZ(S)$ and $G\Omega(S)$ will play a central role in our investigations.

3. Generalized inner automorphisms. We will now introduce a class of automorphisms of an arbitrary semigroup S which in the case of groups reduces exactly to the set of all inner automorphisms. Another generalization of an inner automorphism was introduced by DUBREIL [4] and was intensively studied by CROISOT [3] for cancellative semigroups and by THIERREN [14] for reductive semigroups. We will see in Section 6 by an example that these two generalizations of inner automorphisms of a group are very different.

3. 1 Proposition. *Let S be any semigroup, $(\lambda, \varrho) \in G\Omega(S)$, and assume that λ and ϱ are permutable. Then $\lambda^{-1} \in \Lambda(S)$, λ^{-1} is permutable with ϱ, and the function $\delta_{(\lambda,\varrho)}$ defined by:*

$$s\delta_{(\lambda,\varrho)} = (\lambda^{-1}s)\varrho \quad (s \in S) \tag{2}$$

is an automorphism of S.

Proof. Since both λ and ϱ are permutations on the set S, so is $\delta_{(\lambda,\varrho)}$. For any $x, y \in S$, we obtain $\lambda[(\lambda^{-1}x)y] = (\lambda\lambda^{-1}x)y = xy = (\lambda\lambda^{-1})(xy) = \lambda[\lambda^{-1}(xy)]$ which implies $(\lambda^{-1}x)y = \lambda^{-1}(xy)$. Using this, we compute

$$(xy)\delta_{(\lambda,\varrho)} = [\lambda^{-1}(xy)]\varrho = [(\lambda^{-1}x)y]\varrho = (\lambda^{-1}x)(y\varrho) = (\lambda^{-1}x)[(\lambda\lambda^{-1}y)\varrho] =$$
$$= (\lambda^{-1}x)\{\lambda[(\lambda^{-1}y)\varrho]\} = [(\lambda^{-1}x)\varrho][(\lambda^{-1}x)\varrho] = (x\delta_{(\lambda,\varrho)})(y\delta_{(\lambda,\varrho)})$$

as required. Further, $\lambda[(\lambda^{-1}x)\varrho] = (\lambda\lambda^{-1}x)\varrho = x\varrho = \lambda[\lambda^{-1}(x\varrho)]$ and thus $(\lambda^{-1}x)\varrho = \lambda^{-1}(x\varrho)$.

In view of 3. 1, we may write $x\delta_{(\lambda,\varrho)} = \lambda^{-1}x\varrho$ without ambiguity. If S has an identity element, we may define the *inner automorphism* ε_a induced by an element $a \in G(S)$ by the usual formula:

$$s\varepsilon_a = a^{-1}sa \quad (s \in S). \tag{3}$$

In such a case, $\delta_{(\lambda,\varrho)} = \varepsilon_{e\varrho}$ where e is the identity of S. Conversely, for $a \in G(S)$, we have $\delta_{\pi_a} = \varepsilon_a$. It is then natural to introduce the following notion.

3.2 Definition. With the notation of 3.1, $\delta_{(\lambda,\varrho)}$ is the *generalized inner automorphism* of S induced by (λ, ϱ).

The *group of all automorphisms* of a semigroup S will be denoted by $\mathscr{A}(S)$, the set of *all generalized inner automorphisms* by $\mathscr{I}(S)$. As we have seen above, in the case that S has an identity, $\mathscr{I}(S)$ coincides with the group of inner automorphisms of S. The introduced terminology is further justified by a theorem valid for a large class of semigroups, which reduces to the familiar connection between $\mathscr{I}(S)$ and $C(S)$ when S is a group. For this we need some preliminaries. Recall that S is *weakly reductive* if the mapping $\pi: a \to \pi_a$ $(a \in S)$ is one-to-one (and thus an isomorphism of S onto $\Pi(S)$); S is *globally idempotent* if $S^2 = S$.

3.3 Lemma. *The following statements concerning a semigroup S which is either weakly reductive or globally idempotent are true:*

a) *If* $(\lambda, \varrho), (\lambda', \varrho') \in \Omega(S)$, *then* λ *and* ϱ' *are permutable.*
b) $C\Omega(S) = \{(\lambda, \varrho) \in \Lambda(S) \times P(S) | \lambda s = s\varrho$ *for all* $s \in S\}$.
c) $\pi_{\Lambda|_{C\Omega(S)}}$ *is an isomorphism of* $C\Omega(S)$ *onto* $Z(S)$.

We omit the proof. In a different form, a) is mentioned in CLIFFORD [1].

Under the hypotheses of the lemma, the centroid of S can be identified with the center of $\Omega(S)$, and this case is the most interesting one. In particular $Z(S)$ is then commutative. This represents a slight improvement over ([10], V, § 4, Proposition 1) where $\mathfrak{Z}_r(\mathfrak{A})=0$ or $\mathfrak{Z}_l(\mathfrak{A})=0$ can be replaced by vanishing of the double annihilator $\{a \in \mathfrak{A} | ax = xa = 0$ for all $x \in \mathfrak{A}\}$. Furthermore, part a) shows that $\delta_{(\lambda,\varrho)}$ is defined for every element $(\lambda, \varrho) \in G\Omega(S)$.

The proof of the following lemma is a straightforward verification and is omitted.

3.4. Lemma. *Let θ be an isomorphism of a semigroup S onto a semigroup T. Then the function $\bar{\theta}$ defined by:*

$$\bar{\theta}: (\lambda, \varrho) \to (\bar{\lambda}, \bar{\varrho}) \qquad ((\lambda, \varrho) \in \Omega(S)) \tag{4}$$

where

$$\bar{\lambda}t = [\lambda(t\theta^{-1})]\theta, \quad t\bar{\varrho} = [(t\theta^{-1})\varrho]\theta \qquad (t \in T) \tag{5}$$

is an isomorphism of $\Omega(S)$ onto $\Omega(T)$.

The following is the principal result of this section. Recall the notation (2), (3), (4), (5), $\mathscr{A}(S)$, $\mathscr{I}(S)$.

3.5 Theorem. *Let S be a semigroup which is either weakly reductive or globally idempotent. Then the mapping*

$$\chi: (\lambda, \varrho) \to \delta_{(\lambda,\varrho)} \qquad ((\lambda, \varrho) \in G\Omega(S))$$

is a homomorphism of $G\Omega(S)$ onto $\mathscr{I}(S)$ with kernel $GC\Omega(S)$ so that

$$G\Omega(S)/GC\Omega(S)\cong\mathscr{I}(S).$$

Moreover, $\pi_\Lambda GC\Omega(S)=GZ(S)$ *and* $\bar{\delta}_{(\lambda,\varrho)}=\varepsilon_{(\lambda,\varrho)}$ *for all* $(\lambda,\varrho)\in G\Omega(S)$.

Proof. Using part a) of 3. 3, for any $(\lambda,\varrho),(\varphi,\psi)\in G\Omega(S)$ and $s\in S$, we have

$$s\delta_{(\lambda,\varrho)}\delta_{(\varphi,\psi)}=\varphi^{-1}(\lambda^{-1}s\varrho)\psi=(\lambda\varphi)^{-1}s(\varrho\psi)=s\delta_{(\lambda\varphi,\varrho\psi)}=s\delta_{(\lambda,\varrho)(\varphi,\psi)}$$

and hence χ is a homomorphism. Let $(\lambda,\varrho)\in G\Omega(S)$. Then $(\lambda,\varrho)\in\ker\chi$ if and only if $\delta_{(\lambda,\varrho)}=(\iota_{\Omega(S)},\iota_{\Omega(S)})$, equivalently $\lambda^{-1}s\varrho=s$ for all $s\in S$, which can be written as $s\varrho=\lambda s$ for all $s\in S$. By part b) of 3. 3, the latter is equivalent to $(\lambda,\varrho)\in C\Omega(S)$. Consequently $(\lambda,\varrho)\in\ker\chi$ if and only if $(\lambda,\varrho)\in G\Omega(S)\cap C\Omega(S)=GC\Omega(S)$ as required. The equality $\pi_\Lambda GC\Omega(S)=CZ(S)$ follows from part c) of 3. 3. For $(\lambda,\varrho)\in\in G\Omega(S)$ and $(\varphi,\psi)\in\Omega(S)$, we have

$$(\varphi,\psi)\bar{\delta}_{(\lambda,\varrho)}=(\bar{\varphi},\bar{\psi})$$

where for any $s\in S$,

$$\bar{\varphi}s=[\varphi(s\delta^{-1}_{(\lambda,\varrho)})]\delta_{(\lambda,\varrho)}=\lambda^{-1}[\varphi(\lambda s\varrho^{-1})]\varrho=(\lambda^{-1}\varphi\lambda)s$$

and analogously $s\bar{\psi}=s(\varrho^{-1}\psi\varrho)$, which implies

$$(\bar{\varphi},\bar{\psi})=(\lambda^{-1}\varphi\lambda,\varrho^{-1}\psi\varrho)=(\lambda,\varrho)^{-1}(\varphi,\psi)(\lambda,\varrho)=(\varphi,\psi)\varepsilon_{(\lambda,\varrho)}.$$

Consequently $\bar{\delta}_{(\lambda,\varrho)}=\varepsilon_{(\lambda,\varrho)}$ proving the last assertion of the theorem.

It is clear that for the case when S is a group, the foregoing result reduces to the familiar theorem in group theory. If S is weakly reductive, we may identify $\Pi(S)$ and S; the last assertion of the above theorem then states that the generalized inner automorphisms are the restrictions of inner automorphisms of $\Omega(S)$ to S. In particular, for the case of $S=\mathfrak{F}(\mathfrak{M},\mathfrak{N})$ and $\Omega(S)=\mathfrak{L}(\mathfrak{M},\mathfrak{N})$, with the obvious identifications, where $\mathfrak{M},\mathfrak{N}$ is a pair of dual vector spaces over a (not necessarily commutative) field Φ, the set of all "quasi-inner automorphisms" of MALCEV forms a proper subgroup of $\mathscr{I}(S)$ as shown by ROSENBERG ([13], p. 125).

It should be noted that in general $GC\Omega(S)\neq CG\Omega(S)$. For example, if $S=\mathscr{M}^\circ(G;I,I;\Delta)$ is a Brandt semigroup, it can be shown that the equality fails to occur only in the case when G is trivial and I has exactly 2 elements (stated by HOEHNKE [9], Satz 1), in which case $G\Omega(S)$ is of order 2 and $GC\Omega(S)$ is trivial. In section 6 we will discuss a similar example when S is cancellative. Further properties of the concepts discussed above, as well as the proofs omitted here, can be found in [11].

4. The congruence σ_S. We have seen in the preceding section that for a semigroup S with identity, $Z(S)$ can be identified with $C(S)$. If S is also a group, $GZ(S)=$ $=Z(S)$ and we may consider S as a Schreier extension of $Z(S)$ by $S/Z(S)$. The following will show how this can be generalized to the situation in which S is a semigroup satisfying relatively weak conditions.

4.1 Definition. For any semigroup S, let σ_S be the equivalence relation on S whose classes are the orbits of $GZ(S)$. Hence for any $a, b \in S$, $a\sigma_S b$ if and only if there exists $\lambda \in GZ(S)$ such that $a=\lambda b$.

It is immediate that σ_S is a congruence relation on S. We will often write σ instead of σ_S if there is no danger of confusion. If S has an identity element, then the classes of σ_S coincide with the orbits of $GC(S)$. In particular, for any semigroup S and $a, b \in S$, $\pi_a \sigma_{\Omega(S)} \pi_b$ if and only if $\pi_a=(\lambda, \varrho)\pi_b$ for some $(\lambda, \varrho) \in GC\Omega(S)$, which is in turn equivalent to $\pi_a=\pi_{\lambda b}$ in view of (1). Consequently, for a weakly reductive semigroup S, we have $\pi_a \sigma_{\Omega(S)} \pi_b$ if and only if $a\sigma_S b$. In such a case, if we identify $\Pi(S)$ with S, we may write $\sigma_{\Omega(S)}|_{\Pi(S)}=\sigma_S$.

It follows from the definition of σ that $GZ(S)$ acts as a transitive group of permutations on each σ-class. If $GZ(S)$ acts simply transitively on each σ-class, we will show that S can be expressed as a Schreier extension of $GZ(S)$ by S/σ in the sense of the following (cf. [12])

4.2 Definition. Let Q be a semigroup, Φ be an (additively written) abelian group, and let $[\ ,\ ]: Q\times Q \to \Phi$ be a function satisfying

$$[a, b]+[ab, c] = [a, bc]+[b, c] \qquad (a, b, c \in Q). \tag{6}$$

Let $S = Q\times\Phi$ together with the multiplication

$$(a, \alpha)(b, \beta) = (ab, [a, b]+\alpha+\beta).$$

(It is easy to verify that (6) is equivalent to associativity.) We call S a *Schreier extension of Φ by Q* and denote it by $(Q, \Phi, [\ ,\ ])$.

We are now in a position to prove the desired result. It should now be noted that a sufficient condition on a semigroup S in order that $GZ(S)$ act simply transitively on each σ-class is weak cancellation, viz., the conjunction of $xa=ya$ and $ax=ay$ implies $x=y$. For if $\lambda \in GZ(S)$ and $a=\lambda b=\lambda' b$, then for any $x \in S$, $(\lambda x)a=x(\lambda a)= =x(\lambda' a)=(\lambda' x)a$ and analogously $a(\lambda x)=a(\lambda' x)$, which by weak cancellation yields $\lambda=\lambda'$.

4.3 Theorem. *A semigroup S for which $GZ(S)$ acts simply transitively on each σ-class is isomorphic to a Schreier extension of $GZ(S)$ by S/σ.*

Proof. Let $Q=S/\sigma$ and arbitrarily choose a system $\{z_a\}_{a\in Q}$ of representatives of σ-classes. Letting $\Phi=GZ(S)$, define a function $[\ ,\ ]: Q\times Q \to \Phi$ by the requirement: $[a, b] \in \Phi$ for which $[a, b]z_c=z_a z_b$ where $z_c \sigma z_a z_b$. Next define a function χ on $Q\times\Phi$ by:

$$\chi: x \to (a, \lambda) \quad \text{if} \quad x \in \sigma\text{-class } a \text{ and } \lambda z_a=x.$$

The hypothesis on Φ implies that χ is a bijection of S onto $Q\times\Phi$. For $x\chi=(a,\lambda)$, $y\chi=(b,\mu)$, $(xy)\chi=(c,\nu)$, we obtain

$$(\lambda\mu[a,b])z_c=\lambda\mu\{[a,b]z_c\}=\lambda\mu(z_a z_b)=(\lambda z_a)(\mu z_b)=xy$$

which by simple transitivity implies $\lambda\mu[a,b]=\nu$. Writing Φ additively, we obtain

$$(x\chi)(y\chi)=(a,\lambda)(b,\mu)=(ab,[a,b]+\lambda+\mu)=(xy)\chi$$

which shows that χ is a homomorphism. Condition (6) on $[\ ,\ ]$ is equivalent to associativity and hence follows here from the associativity in S. Therefore χ is an isomorphism of S onto $(Q,\Phi,[\ ,\ ])$ as required.

If S has an identity element 1, then 1 can be chosen as the representative of its σ-class e, which then yields $[a,e]=[e,a]=0$ where 0 in additive notation stands for the identity function. This is the usual "initial condition" imposed on "Schreier extensions" in group, ring, or semigroup theory (see RÉDEI [12]). Conversely, if $S=(Q,\Phi,[\ ,\ ])$, e is the identity of Q, and $[a,e]=[e,a]=0$ for all $a\in Q$, then $(e,0)$ is the identity of S. In such a case, the mapping $\alpha\to(e,\alpha)$ $(\alpha\in\Phi)$ embeds Φ into S in a natural way. For the Schreier extensions of semigroups S and S' constructed above, we have the following isomorphism criterion.

4.4 Theorem. *Let S and S' satisfy the condition in* 4.3, *and let $T=(Q,\Phi,[\ ,\ ])$, $T'=(Q',\Phi',[\ ,\ ]')$ be the isomorphic copies of S and S', respectively, as in* 4.3. *Then $S\cong S'$ if and only if there exists an isomorphism θ of Q onto Q' and for each $a\in Q$ there is a bijection η_a of Φ onto Φ' such that*

$$\eta_a\alpha+\eta_b\beta+[a\theta,b\theta]'=\eta_{ab}([a,b]+\alpha+\beta)\qquad(a,b\in Q,\quad\alpha,\beta\in\Phi).\tag{7}$$

Proof. Suppose first that $S\cong S'$. Then there exists an isomorphism ψ of T onto T'. It is easy to see that $(a,\alpha)\sigma_T(b,\beta)$ if and only if $a=b$. It follows from 3.4 that an isomorphism preserves σ-classes. Consequently there exists an isomorphism θ of Q onto Q' making the diagram

$$\begin{tikzcd}
T \arrow[r, "\psi"] \arrow[d, "\varphi"'] & T' \arrow[d, "\varphi'"] \\
T/\sigma_T \arrow[d, "\xi"'] & T'/\sigma_{T'} \arrow[d, "\xi'"] \\
Q \arrow[r, "\theta"] & Q'
\end{tikzcd}$$

commutative, where φ and φ' are canonical homomorphisms, and $\langle a,\alpha\rangle\xi=a$, $\langle a',\alpha'\rangle\xi'=a'$ for σ-classes $\langle a,\alpha\rangle$ and $\langle a',\alpha'\rangle$ of T and T', respectively. Hence for any $(a,\alpha)\in T$ and $(a,\alpha)\psi=(b,\beta)$, we obtain $(a,\alpha)\varphi\xi\theta=\langle a,\alpha\rangle\xi\theta=a\theta$ and $(a,\alpha)\psi\varphi'\xi'=$

$=(b, \beta)\varphi'\xi'=\langle b, \beta\rangle\xi'=b$, and thus $a\theta=b$. Now writing β in the form $\eta_a\alpha$, we see that

$$(8) \qquad (a, \alpha)\psi=(a\theta, \eta_a\alpha) \qquad (a\in Q, \quad \alpha\in\Phi)$$

where η_a is a mapping of Φ into Φ' for every $a\in Q$. Thus for any (a, α), $(b, \beta)\in T$, we have

$$(a, \alpha)\psi(b, \beta)\psi = (a\theta, \eta_a\alpha)(b\theta, \eta_b\beta) = \big((a\theta)(b\theta), [a\theta, b\theta]'+\eta_a\alpha+\eta_b\beta\big),$$

$$[(a, \alpha)(b, \beta)]\psi = (ab, [a, b]+\alpha+\beta)\psi = \big((ab)\theta, \eta_{ab}([a, b]+\alpha+\beta)\big)$$

which implies (7). For any $a\in Q$ and $\alpha'\in\Phi'$ there exists $\alpha\in\Phi$ such that $(a, \alpha)\psi= =(a\theta, \alpha')$ since both ψ and θ are onto. By (8), we must have $\eta_a\alpha=\alpha'$ proving that η maps Φ onto Φ'. If $\eta_a\alpha=\eta_a\beta$, then by (8), we have $(a, \alpha)\psi=(a, \beta)\psi$ so that $\alpha=\beta$ since ψ is one-to-one. Thus η_a is a bijection.

Conversely, if the functions θ and η_a are given as in the theorem, then ψ given by (8) is easily seen to be an isomorphism of T onto T', which in turn implies the existence of an isomorphism of S onto S'.

We have seen in the proof that the converse is valid without $Q=S/\sigma_S$, $Q'=S'/\sigma_{S'}$. Hence for arbitrary Schreier extensions $T=(Q, \Phi, [\ ,\ ])$ and $T'=(Q', \Phi', [\ ,\ ]')$, 4. 4 furnishes sufficient conditions for their isomorphism. However, they are in general not necessary, for it could happen that $T\cong T'$ but $Q\not\cong Q'$. Furthermore, in the case of 4. 4, Φ and Φ' are isomorphic which is not explicit in these conditions.

5. A dense embedding. We will now establish that for a left or right reductive globally idempotent semigroup Q, a Schreier extension of an abelian group Φ by Q can be embedded (and in some special way) into several semigroups which are wreath products of a semigroup of transformations on Q and Φ. First we recall the pertinent definitions.

If B is an ideal of a semigroup A, then A is an *ideal extension* of B; if in addition the equality congruence on A is the only congruence on A whose restriction to B is the equality congruence on B, then A is a *dense extension* of B; if also A is, under inclusion, a maximal dense extension of A, then B is a *densely embedded ideal* of A. A subsemigroup C of A is a *densely embedded subsemigroup* if C is a densely embedded ideal of its idealizer $i_A(C)$ in A. An isomorphism φ of a semigroup D into A is a *dense embedding* if $D\varphi$ is a densely embedded subsemigroup of A, and we say that D can be *densely embedded* in A. Using the concepts of a left ideal, left idealizer, etc., one defines analogously an *l-densely embedded* ideal, subsemigroup, embedding etc. For an extensive discussion concerning these concepts, see GLUSKIN [5], [6].

Let X be a nonempty set, P a semigroup of transformations on X, written on the left, and let G be a group. On $S = P \times G$ define a multiplication by

$$(\alpha, \varphi)(\alpha', \varphi') = (\alpha\alpha', \varphi^{\alpha} \cdot \varphi')$$

where $(\varphi^{\alpha} \cdot \varphi')x = (\varphi\alpha x)(\varphi' x)$ $(x \in X)$. Then S is a semigroup called the *(left) wreath product* of P and G and will be denoted by $PwlG$.

The following discussion is motivated by ([6], Section 1).

Let Q be a right reductive semigroup, i.e., $ax=bx$ for all $x \in Q$ implies $a=b$, let Φ be an abelian group, and let $S=(Q, \Phi, [\ ,\ ])$ be a Schreier extension of Φ by Q.

Let $\lambda \in \Lambda(S)$; then for any $(a, \alpha) \in S$ we have

$$\lambda(a, \alpha) = \big(\xi(a, \alpha), \theta(a, \alpha)\big) \tag{9}$$

for some functions ξ and θ. Hence

$$[\lambda(a, \alpha)](b, \beta) = \big(\xi(a, \alpha), \theta(a, \alpha)\big)(b, \beta) = \big(\xi(a, \alpha)b, [\xi(a, \alpha), b] + \theta(a, \alpha) + \beta\big),$$

$$\lambda[(a, \alpha)(b, \beta)] = \lambda(ab, [a, b] + \alpha + \beta) = \big(\xi(ab, [a, b] + \alpha + \beta), \theta(ab, [a, b] + \alpha + \beta)\big),$$

and thus

$$\xi(a, \alpha)b = \xi(ab, [a, b] + \alpha + \beta), \tag{10}$$

$$[\xi(a, \alpha), b] + \theta(a, \alpha) + \beta = \theta(ab, [a, b] + \alpha + \beta). \tag{11}$$

We substitute β in (10) by $\beta - [a, b] - \alpha$ and obtain $\xi(a, \alpha)b = \xi(ab, \beta)$. Since this is true for all $\alpha, \beta \in \Phi$, we also have $\xi(a, \beta)b = \xi(ab, \beta)$, and thus $\xi(a, \alpha)b = \xi(a, \beta)b$ for all $b \in Q$. By right reductivity of Q, we conclude that $\xi(a, \alpha) = \xi(a, \beta)$, i.e., $\xi(a, \alpha)$ is independent of α and we may write ξa instead of $\xi(a, \alpha)$. But then (10) yields $(\xi a)b = \xi(ab)$ so that $\xi \in \Lambda(Q)$.

Now (11) implies

$$\theta(a, \alpha) = \theta(ab, [a, b] + \alpha + \beta) - [\xi a, b] - \beta \tag{12}$$

which for $\beta = -[a, b] - \alpha$ becomes

$$\theta(a, \alpha) = \theta(ab, 0) - [\xi a, b] + [a, b] + \alpha. \tag{13}$$

For $\alpha = 0$, (11) takes on the form

$$\theta(a, 0) = \theta(ab, 0) - [\xi a, b] + [a, b]. \tag{14}$$

Now let $\eta a = \theta(a, 0)$ for all $a \in Q$. Substituting $\theta(ab, 0)$ from (14) into (13) in the new notation we have

$$\theta(a, \alpha) = \eta a + \alpha \qquad ((a, \alpha) \in S). \tag{15}$$

Further, (15) substituted into (12) yields $\eta a+\alpha = \eta(ab)+[a, b]+\alpha+\beta-[\xi a, b]-\beta$ and thus η satisfies the condition

$$\eta a-\eta(ab) = [a, b]-[\xi a, b] \qquad (a, b\in Q). \tag{16}$$

Hence (9) becomes

$$\lambda(a, \alpha) = (\xi a, \eta a+\alpha) \qquad ((a, \alpha)\in S). \tag{17}$$

5. 1 Theorem. *With the notation introduced, define a function* ψ *by:*

$$\psi:\lambda\to(\xi, \eta) \qquad (\lambda\in\Lambda(S)) \tag{18}$$

where $\xi\in\Lambda(Q)$ *and* $\eta\in G^Q$ *satisfy* (16), *and* λ *satisfies* (17). *Then* ψ *embeds* $\Lambda(S)$ *into* $\Lambda(Q)wl\Phi$. *In addition, if* Q *is globally idempotent, then* $\Lambda(S)\psi$ *is the largest subsemigroup of* $\Lambda(Q)wl\Phi$ *containing* $\Gamma(S)\psi$ *as a left ideal.*

Proof. If $\lambda\psi=(\xi, \eta)$ and $\lambda'\psi=(\xi', \eta')$, then for any $(a, \alpha)\in S$, $\lambda\lambda'(a, \alpha)= = \lambda(\xi' a, \eta' a+\alpha) = (\xi\xi' a, \eta\xi' a+\eta' a+\alpha)$ which in the multiplicative notation can be written as $\lambda\lambda'(a, \alpha) = (\xi\xi' a, (\eta^{\xi'}\cdot\eta')a+\alpha)$. Hence $(\lambda\lambda')\psi = (\xi\xi', \eta^{\xi'}\cdot\eta') = = (\xi, \eta)(\xi', \eta') = (\lambda\psi)(\lambda'\psi)$. Thus ψ is a homomorphism, and is clearly one-to-one.

If $(\xi, \eta)\in\Lambda(Q)wl\Phi$ satisfies (16), then λ defined by (17) has the property:

$$\begin{aligned}[\lambda(a, \alpha)](b, \beta) &= (\xi a, \eta a+\alpha)(b, \beta) = ((\xi a)b, [\xi a, b]+\eta a+\alpha+\beta)\\ &= (\xi(ab), \eta(ab)+[a, b]+\alpha+\beta) = \lambda(ab, [a, b]+\alpha+\beta)\\ &= \lambda[(a, \alpha)(b, \beta)].\end{aligned}$$

Thus $\lambda\in\Lambda(S)$ and furthermore $\lambda\psi=(\xi, \eta)$. Consequently

$$\Lambda(S)\psi=\{(\xi, \eta)|\xi \text{ and } \eta \text{ satisfy (16)}\}. \tag{19}$$

Next let $(a, \alpha)\in S$ and $\lambda_{(a, \alpha)}\psi=(\xi, \eta)$. Then for any $(b, \beta)\in S$,

$$\lambda_{(a, \alpha)}(b, \beta) = (a, \alpha)(b, \beta) = (ab, [a, b]+\alpha+\beta) \tag{20}$$

and on the other hand,

$$\lambda_{(a, \alpha)}(b, \beta) = (\xi b, \eta b+\beta). \tag{21}$$

Comparing (20) and (21), we obtain $\xi=\lambda_a$, $\eta b = [a, b]+\alpha$ for all $b\in Q$. Conversely, for the pair (λ_a, η) with $\eta b = [a, b]+\alpha$ for all $b\in Q$, we obtain

$$\eta b-\eta(bc) = [a, b]+\alpha-[a, bc]-\alpha = [b, c]-[ab, c] = [b, c]-[\lambda_a b, c]$$

using (6). Consequently

$$\Gamma(S)\psi = \{(\lambda_a, \eta)|\eta b = [a, b]+\alpha \text{ for some } \alpha\in\Phi\}.$$

Suppose now that Q is globally idempotent. Since $\Gamma(S)$ is a left ideal of $\Lambda(S)$, we have that $\Gamma(S)\psi$ is a left ideal of $\Lambda(S)\psi$. Now suppose that $(\sigma, \tau) \in \Lambda(S) wl \Phi$ has the property

$$(\sigma, \tau)(\xi, \eta) \in \Gamma(S)\psi \quad \text{for all} \quad (\xi, \eta) \in \Gamma(S)\psi.$$

It follows that (σ, τ) induces a left translation on $\Gamma(S)\psi$, and thus by the isomorphism ψ, there exists $\lambda \in \Lambda(S)$ such that $\lambda\psi$ and (σ, τ) have the same effect upon $\Gamma(S)\psi$. Writing $\lambda\psi = (\xi, \eta)$, we then have

$$(\xi, \eta)(\lambda_a, \theta) = (\sigma, \tau)(\lambda_a, \theta) \qquad ((\lambda_a, \theta) \in \Gamma(S)\psi).$$

Hence $\xi\lambda_a = \sigma\lambda_a$, $\eta^{\lambda_a} \cdot \theta = \tau^{\lambda_a} \cdot \theta$ so that $\xi\lambda_a x = \sigma\lambda_a x$, $\eta\lambda_a x + \theta x = \tau\lambda_a x + \theta x$ and thus $(\xi a)x = (\sigma a)x$, $\eta(ax) = \tau(ax)$ $(a, x \in Q)$. Right reductivity of Q implies $\xi = \sigma$, and $Q^2 = Q$ implies $\eta = \tau$. Thus $(\xi, \eta) = (\sigma, \tau)$ which proves that $(\sigma, \tau) \in \Lambda(S)\psi$. Therefore $\Lambda(S)\psi$ is the largest subsemigroup of $\Lambda(Q) wl \Phi$ containing $\Gamma(S)\psi$ as a left ideal.

5.2 Lemma. *For a right reductive semigroup S, $i_{\Lambda(S)}(\Gamma(S)) = \tilde{\Lambda}(S)$ and $\Gamma(S)$ is a densely embedded ideal of $\tilde{\Lambda}(S)$.*

Proof. Right reductivity of S implies that π_Λ is an isomorphism of $\Omega(S)$ onto $\tilde{\Lambda}(S)$ mapping $\Pi(S)$ onto $\Gamma(S)$. Since $\Pi(S)$ is a densely embedded ideal of $\Omega(S)$ by ([6], 1. 3. 5), $\Gamma(S)$ must be a densely embedded ideal of $\tilde{\Lambda}(S)$. In particular, $\tilde{\Lambda}(S) \subseteqq i_{\Lambda(S)}(\Gamma(S))$. If $\lambda \in i_{\Lambda(S)}(\Gamma(S))$, then for every $a \in S$, there exists $a' \in S$ such that $\lambda_a \lambda = \lambda_{a'}$, where a' is unique by right reductivity of S. Define ϱ on S by the requirement $\lambda_a \lambda = \lambda_{a\varrho}$ $(a \in S)$. Then

$$\lambda_{(ab)\varrho} = \lambda_{ab}\lambda = \lambda_a(\lambda_b\lambda) = \lambda_a\lambda_{b\varrho} = \lambda_{a(b\varrho)},$$

$$\lambda_{(a\varrho)b} = (\lambda_{a\varrho})\lambda_b = (\lambda_a\lambda)\lambda_b = \lambda_a(\lambda\lambda_b) = \lambda_a\lambda_{\lambda b} = \lambda_{a(\lambda b)}$$

so that $(\lambda, \varrho) \in \Omega(S)$. Consequently $\lambda \in \tilde{\Lambda}(S)$ proving that $i_{\Lambda(S)}(\Gamma(S)) \subseteqq \tilde{\Lambda}(S)$.

5.3 Corollary. *For a right reductive semigroup Q, isomorphism* (18) *induces an embedding of S into $\Gamma(Q) wl \Phi$, and if $Q^2 = Q$, it also induces an l-dense and dense embedding of S into $\Lambda(Q) wl \Phi$.*

Proof. It suffices to compose the isomorphism $a \to \lambda_a$ with ψ and apply 5. 1, 5. 2 and ([6], 1. 3. 5 and 1. 10. 2).

5.4 Corollary. *Every right reductive semigroup S for which $GZ(S)$ acts simply transitively on each σ-class can be embedded into $\Gamma(S/\sigma) wl GZ(S)$. If S is also globally idempotent, then S can also be l-densely and densely embedded into $\Lambda(S/\sigma) wl GZ(S)$.*

Proof. Apply 4. 3 and 5. 3.

6. Cancellative semigroups. For the class of cancellative semigroups, we are able to prove much stronger statements concerning σ_S than in the general case. Throughout this section S denotes an arbitrary cancellative semigroup.

6. 1 Proposition. *For any $a, b \in S$, we have $a\sigma b$ if and only if*

(22) $$aS=bS, \quad Sa=Sb, \quad axb=bxa \quad \textit{for all} \quad x \in S.$$

Proof. First let $a=\lambda b$ where $\lambda \in GZ(S)$. For any $x \in S$, we obtain $ax=(\lambda b)x=$ $=b(\lambda x) \in bS$, $xa=x(\lambda b)=(\lambda x)b \in Sb$ proving $aS \subseteqq bS$ and $Sa \subseteqq Sb$. By symmetry, we also have $bS \subseteqq aS$ and $Sb \subseteqq Sa$. Further, $axb=ax(\lambda a)=(\lambda a)xa=bxa$, and thus the pair a, b satisfies (22).

Conversely, let $a, b \in S$ satisfy (22). For every $x \in S$ there exists a unique $y \in S$ such that $ax=by$. We then define a function λ on S by the requirement $ax=b(\lambda x)$ $(x \in S)$. Similarly define λ' by $bx=a(\lambda' x)$ $(x \in S)$. Then $ax=b(\lambda x)=a(\lambda' \lambda x)$ so that $x=\lambda' \lambda x$ and similarly $x=\lambda \lambda' x$ for all $x \in S$, which shows that λ is invertible. Furthermore, λ is obviously a left translation of S. Analogously define ϱ on S by $xa=$ $=(x\varrho)b$; a dual proof shows that ϱ is an invertible right translation on S. For any $x \in S$, we also have $b(\lambda x)b=axb=bxa=b(x\varrho)b$ so that $\lambda x=x\varrho$. But then part b) of 3. 3 implies that $(\lambda, \varrho) \in C\Omega(S)$, and hence $\lambda \in GZ(S)$. Finally $ax=b(\lambda x)$ implies $ax=(\lambda b)x$ so that $a=\lambda b$.

6. 2 Corollary. *Let S be a cancellative semigroup without idempotents. Then there exists a nontrivial group G and a cancellative semigroup V which is an ideal extension of S by G^0 such that $G \subseteqq C(V)$ if and only if there exist distinct elements a and b of S for which $aS=bS$, $Sa=Sb$, $axb=bxa$ for all $x \in S$.*

Proof. The last condition is equivalent to the statement that σ_S is not the equality relation on S, which is in turn equivalent to the assertion that $GZ(S)$ is nontrivial. Now if $V = S \cup G$ is an extension of S described above, then V is a dense extension of S and the canonical homomorphism $G \to \Omega(S)$ provides an isomorphism of G into $GC\Omega(S)$ (see [7] for a general discussion). Hence if G is nontrivial, so is $GC\Omega(S)$ and thus also $GZ(S)$. Conversely, if $GZ(S)$ is nontrivial, then $V = S \cup GC\Omega(S)$ with the identification of S and $\Pi(S)$, provides an extension of S of the desired form.

For the case of a commutative cancellative S, this corollary reduces to ([8], Theorem 4. 4). The next lemma is also of independent interest.

6. 3 Lemma. *A dense extension of a left cancellative right reductive semigroup is left cancellative and right reductive.*

Proof. Let V be a dense extension of a left cancellative right reductive semigroup S. Let $a, b \in V$ and suppose that $as=bs$ for all $s \in S$. In particular $(ta)s=$

$=(tb)s$ for all $t, s \in S$ where $ta, tb \in S$, so by right reductivity of S we obtain $ta=tb$. Consequently $as=bs$, $sa=sb$ for all $s \in S$, which by ([7], Theorem 3. 7) implies $a=b$ since V is a dense extension of S. In particular, V is right reductive. Suppose next that $ca=cb$ for some $a, b, c \in V$. Then $(sc)(at)=(sc)(bt)$ for all $s, t \in S$, where $sc, at, bt \in S$. Hence the left cancellation in S yields $at=bt$ for all $t \in S$. But then $a=b$ as we have seen above. Thus V is left cancellative.

6. 4 Corollary. *If S is cancellative, so is $\Omega(S)$.*

Proof. This follows from 6. 3, its dual, and the fact that $\Omega(S)$ is a (maximal) dense extension of $\Pi(S) \cong S$ by ([6], 1. 3. 5).

Let S be a cancellative semigroup, and for every $a \in S$ denote by a^* the σ-class containing a. Let $a\sigma b$ and $(\lambda, \varrho) \in \Omega(S)$. Then $a=\varphi b$ for some $\varphi \in GZ(S)$ which implies $\lambda a=\lambda\varphi b=\varphi(\lambda b)$ and $a\varrho=(\varphi b)\varrho=\varphi(b\varrho)$, so that $\lambda a\sigma\lambda b$ and $a\varrho\sigma b\varrho$. This makes it possible to define the function θ below.

6. 5 Theorem. *For a cancellative semigroup S, define a function θ by:*

$$\theta\colon (\lambda, \varrho) \to (\bar{\lambda}, \bar{\varrho}) \quad ((\lambda, \varrho) \in \Omega(S)) \tag{23}$$

where

$$\bar{\lambda}a^*=(\lambda a)^*, \; a^*\bar{\varrho}=(a\varrho)^* \quad (a \in S). \tag{24}$$

Then θ is a homomorphism of $\Omega(S)$ into $\Omega(S/\sigma_S)$ and $\ker\theta=\sigma_{\Omega(S)}$. *Moreover $\Omega(S/\sigma_S)$ is cancellative.*

Proof. The discussion before the theorem shows that both $\bar{\lambda}$ and $\bar{\varrho}$ are single-valued. For any $a, b \in S$, we obtain

$$(\bar{\lambda}a^*)b^*=(\lambda a)^*b^*=[(\lambda a)b]^*=[\lambda(ab)]^*=\bar{\lambda}(ab)^*=\bar{\lambda}(a^*b^*)$$

so $\bar{\lambda} \in \Lambda(S/\sigma)$, and dually $\bar{\varrho} \in P(S/\sigma)$; that $\bar{\lambda}$ and $\bar{\varrho}$ are linked is verified in a similar manner. Thus θ maps $\Omega(S)$ into $\Omega(S/\sigma)$. For (λ, ϱ), $(\varphi, \psi) \in \Omega(S)$ and $a \in S$, we have

$$(\bar{\lambda}\bar{\varphi})a^*=\bar{\lambda}(\bar{\varphi}a^*)=\bar{\lambda}(\varphi a)^*=(\lambda\varphi a)^*=\overline{\lambda\varphi}\,a^*$$

so that $\bar{\lambda}\bar{\varphi}=\overline{\lambda\varphi}$ and dually $\bar{\varrho}\bar{\psi}=\overline{\varrho\psi}$, showing that θ is a homomorphism.

Next let (λ, ϱ), $(\varphi, \psi) \in \Omega(S)$. Then $(\lambda, \varrho)\theta=(\varphi, \psi)\theta$ is successively equivalent to $\bar{\lambda}=\bar{\varphi}$, $\bar{\varrho}=\bar{\psi}$, and to

$$(\lambda a)^*=(\varphi a)^*, \quad (a\varrho)^*=(a\psi)^* \qquad (a \in S),$$

and to

$$\lambda a\sigma\varphi a, \quad a\varrho\sigma a\psi \qquad (a \in S). \tag{25}$$

Suppose that $\lambda a\sigma\varphi a$ for all $a \in S$. By 6. 1, we have

$$(\lambda a)S=(\varphi a)S, \quad S(\lambda a)=S(\varphi a), \quad (\lambda a)x(\varphi a)=(\varphi a)x(\lambda a) \qquad (x \in S).$$

For a fixed $a \in S$, as in the proof of 6. 1, we may define γ and δ by:

$$(\lambda a)x=(\varphi a)(\gamma x), \quad x(\lambda a)=(x\delta)(\varphi a) \qquad (x \in S).$$

It follows as in the proof of 6. 1 that $(\gamma, \delta) \in CG\Omega(S)$ and $\lambda a=\gamma\varphi a$. Further $(a\varrho)a= =a(\lambda a)=a(\gamma\varphi a)=(a\delta\psi)a$ so $a\varrho=a\delta\psi$. For any $b \in S$ we then have $a(\lambda b)=(a\varrho)b= =(a\delta\psi)b=a(\gamma\varphi b)$ and thus $\lambda b=\gamma\varphi b$. As in the preceding step, this also implies $b\varrho=b\delta\psi$. Consequently $(\lambda, \varrho)=(\gamma, \delta)(\varphi, \psi)$ with $(\gamma, \delta) \in GC\Omega(S)$ and hence

$$(\lambda, \varrho)\sigma_{\Omega(S)}(\varphi, \psi).$$

Note that we have used only the first half of (25) and for a single a. Hence $\ker \theta \subseteq \subseteq \sigma_{\Omega(S)}$.

Conversely, suppose that $(\lambda, \varrho)\sigma_{\Omega(S)}(\varphi, \psi)$. Then for some $(\gamma, \delta) \in GC\Omega(S)$, we have $(\lambda, \varrho)=(\gamma, \delta)(\varphi, \psi)$, and thus $\lambda=\gamma\varphi$, $\varrho=\delta\psi$. Thus for every $a \in S$, we obtain $\lambda a=\gamma(\varphi a)$ and $a\varrho=(a\delta)\psi=(\gamma a)\psi=\gamma(a\psi)$, and hence (25) holds since $\gamma \in GZ(S)$. We have seen above that this is equivalent to $(\lambda, \varrho)\theta=(\varphi, \psi)\theta$. Consequently $\sigma_{\Omega(S)} \subseteq \subseteq \ker \theta$, and the equality prevails.

Suppose that $a^*c^*=b^*c^*$. Then $ac\,\sigma\, bc$ and thus $ac=\lambda(bc)$ for some $\lambda \in GZ(S)$. But then $ac=(\lambda b)c$ so that $a=\lambda b$ and thus $a^*=b^*$. It follows that right cancellation in S implies right cancellation in S/σ. By symmetry, we conclude that S/σ is cancellative, which by 6. 4 implies that $\Omega(S/\sigma)$ is cancellative.

The next result shows that for every element b of a cancellative semigroup S we can define a new multiplication on a subset of S in such a way as to make it a semigroup isomorphic with $Z(S)$ and for which b acts as the identity element. The group of units of this semigroup, as a set, coincides with the σ_S-class of b.

6. 6 Theorem. *Let S be a cancellative semigroup. For any $b \in S$, let*

$$\Sigma_b=\{a \in S \mid aS \subseteq bS, \quad Sa \subseteq Sb, \quad axb=bxa \quad \text{for all} \quad x \in S\},$$

and on Σ_b define multiplications $*$ and $\circ$ by the formulae:

$$aa' = b(a * a') = (a \circ a')b.$$

Then Σ_b is closed under $*$, the two multiplications coincide, and the mapping ψ defined by $\psi:\lambda \to \lambda b$ $(\lambda \in Z(S))$ is an isomorphism of $Z(S)$ onto $(\Sigma_b, *)$. In $(\Sigma_b, *)$, b is the identity element and $(b^*, *)=G(\Sigma_b, *) \cong GZ(S)$.

Proof. For $\lambda \in Z(S)$ and any $x \in S$, we obtain $(\lambda b)x=b(\lambda x) \in bS$, $x(\lambda b)= =(\lambda x)b \in Sb$, $(\lambda b)xb=bx(\lambda b)$ which shows that $\lambda b \in \Sigma_b$. If $a \in \Sigma_b$, then similarly as in the second part of the proof of 6. 1, we may show that λ defined by $ax=b(\lambda x)$ $(x \in S)$ has the properties $\lambda \in Z(S)$ and $a=\lambda b$. Thus ψ maps $Z(S)$ onto Σ_b. If $\lambda, \lambda' \in$

$\in Z(S)$ and $\lambda b=\lambda' b$, then for any $x\in S$, $(\lambda x)b=x(\lambda b)=x(\lambda' b)=(\lambda' x)b$ so that $\lambda=\lambda'$ and ψ is one-to-one. For $\lambda, \lambda'\in Z(S)$, we further have

$$b[(\lambda b)*(\lambda' b)] = (\lambda b)(\lambda' b) = b(\lambda\lambda' b)$$

so that $(\lambda\psi)*(\lambda'\psi) = (\lambda\lambda')\psi$ showing that ψ is a homomorphism. Therefore ψ is an isomorphism of $Z(S)$ onto Σ_b, which in particular implies that Σ_b is closed under $*$. For any $a, a'\in\Sigma_b$, we also have

$$b(a*a')b = aa'b = ba'a = b(a'\circ a)b$$

and hence $a*a' = a'\circ a$. But the isomorphism $Z(S)\cong(\Sigma_b, *)$ shows that $a*a' = a'*a$, and we conclude that $*$ and $\circ$ coincide. It is immediate that b is the identity of $(\Sigma_b, *)$, and a comparison with the definition of σ_S quickly shows that the last assertion of the theorem is correct.

Example. Consider the set $S=\{(a, b)|0<a<1, b \text{ real}\}$ under the multiplication $(a, b)(c, d) = (ac, bc+d)$. The mapping

$$(a, b)\to\begin{bmatrix} a & 0\\ b & 1\end{bmatrix}$$

is easily seen to be an isomorphism of S into the multiplicative group of 2×2 non-singular matrices over reals. Thus S is a cancellative semigroup. A straightforward calculation shows that the translational hull of S can be identified with the semigroup

$$T=\{(a, b)|0<a\leqq 1,\ b \text{ real}\}$$

with elements of T acting on elements of S and multiplying among themselves by the same rule as in S. One further verifies easily that

(i) $G(T)=CG(T)=\{(1, b)|b \text{ real}\}$

(ii) $GZ(S)=Z(S)\cong GC(T)=C(T)=\{(1, 0)\}$.

In particular, $GC\Omega(S)$ is trivial while $CG\Omega(S)=G\Omega(S)=C\Omega(S)$ is isomorphic to the group of additive real numbers.

In [4] (Chapitre II, § 4) Dubreil defines an "inner automorphism" of a cancellative semigroup S as follows. For $a\in S$ such that $aS=Sa$, define α_a and β_a by the formulae: $ax=(\alpha_a x)a$, $xa=a(\beta_a x)$ $(s\in S)$. Then α_a and β_a are called inner automorphisms of the first and second category, respectively. It is easy to see that for any cancellative semigroup S and $(\lambda, \varrho)\in G\Omega(S)$, $a\in S$, we have $aS=Sa$ and $\alpha_a=\delta_{(\lambda,\varrho)}$ if and only if $\lambda a\in C(S)$. In the above example,

(iii) $(a, b)S=S(a, b)$ for all $(a, b)\in S$,

(iv) $C(S)=\emptyset$.

Hence S has inner automorphisms of the first (and thus also of the second) category but none is a generalized inner automorphism in our sense. In the case of groups, however, both of these notions reduce to inner automorphisms. For further properties of this type of example, see ([1], § 2. 1, exerc. 9).

7. Commutative cancellative semigroups. For a semigroup S of this class, we can give much more precise and complete information concerning the representation of S as a Schreier extension of $GZ(S)$ by S/σ_S. We start with some auxiliary results.

7. 1 Lemma. *A dense extension V of a commutative reductive semigroup S is commutative.*

Proof. Let $a, b \in V$; then for any $s, t \in S$, we have $s(ab)t=(sa)(bt)=(bt)(sa)=$ $=b(sa)t=bs(at)=(bat)s=s(ba)t$ which by reductivity in S yields $sab=sba$. Since this holds for all $s \in S$, ([7], Theorem 3. 7) implies $ab=ba$ by density of the extension.

In fact, the above V is also reductive, which we will not need here.

7. 2 Corollary. *If S is commutative and cancellative, so is $\Omega(S)$.*

Proof. This follows from 7. 1 and ([6], 1. 3. 5).

7. 3. Lemma. *For any commutative semigroup S we have $Z(S)=\tilde{\Lambda}(S)=\Lambda(S)$.*

Proof. If $\lambda \in \Lambda(S)$, then letting $s\varrho=\lambda s$ $(s \in S)$, we obtain $(\lambda, \varrho) \in \Omega(S)$, which shows that $\Lambda(S) \subseteqq \tilde{\Lambda}(S)$. The inclusion $\Lambda(S) \subseteqq Z(S)$ follows immediately from commutativity.

We infer that for a commutative reductive semigroup, the projection π_Λ furnishes an isomorphism of $\Omega(S)$ onto $\Lambda(S)$, and both of these are commutative. In order to simplify our statements, we introduce the following concept.

7. 4. Definition. A semigroup S is *basic* if S is commutative, cancellative, and $aS=bS$ implies $a=b$.

In view of commutativity and 6. 1, the last condition is equivalent to σ_S being the equality relation. Note that in general $\sigma_{S/\sigma}$ need not be the equality relation, it suffices to take a group G for which $G/C(G)$ has a nontrivial center. For the semigroups under consideration, we have

7. 5. Proposition. *Let S be a commutative cancellative semigroup. Then σ_S is the smallest congruence τ on S for which S/τ is basic.*

Proof. At the end of the proof of 6. 5, we have seen that cancellation in S implies cancellation in S/σ_S. Next suppose that $a^* \sigma_{S^*} b^*$ where $a \to a^*$ is the canonical homomorphism of S onto $S/\sigma=S^*$. By 6. 1, we have $a^* S^*=b^* S^*$. Thus for every $x \in S$ there exists $y \in S$ such that $a^* x^*=b^* y^*$. But then $(ax)^*=(by)^*$ which implies $axS=byS$. In particular, there exists $z \in S$ for which $axy=byz$, and thus $ax=bz$.

This shows that $aS \subseteqq bS$; by symmetry we conclude that $aS = bS$, i.e., $a\sigma_S b$. We have proved that $a^* \sigma_{S*} b^*$ implies $a^* = b^*$, and S/σ_S is basic.

Next let τ be any congruence on S for which S/τ is basic, and let $a \to \hat{a}$ be the canonical homomorphism of S onto $S/\tau = \hat{S}$. If $a\sigma_S b$, then $aS = bS$ so $\hat{a}\hat{S} = \hat{b}\hat{S}$, and since $\hat{S}$ is basic, it follows that $\hat{a} = \hat{b}$. Hence $a\tau b$ proving that $\sigma_S \subseteqq \tau$.

We come now to the principal theorem of this section. It is the culmination of the effort to construct commutative cancellative semigroups out of commutative cancellative semigroups having some special properties and using $GZ(S)$ and S/σ.

7.6 Theorem. *Let Q be a basic semigroup, Φ be an abelian group, $S = (Q, \Phi, [\ ;\ ])$ be a Schreier extension of Φ by Q, and suppose that*

$$[a, b] = [b, a] \qquad (a, b \in Q). \tag{26}$$

Then S is a commutative cancellative semigroup for which $S/\sigma_S \cong Q$, and $GZ(S) \cong \Phi$. Conversely, every commutative cancellative semigroup S is isomorphic to $(Q, \Phi, [\ ,\ ])$ for some basic semigroup Q, abelian group Φ, and a function $[\ ,\ ]$ satisfying (6) *and* (26).

Proof. Let S be as in the first part of the theorem. A simple calculation shows that S is both commutative and cancellative. The mapping $\psi: (a, \alpha) \to a$ for all $(a, \alpha) \in S$, is obviously a homomorphism of S onto Q. Let $(a, \alpha)\sigma_S(b, \beta)$. Then $(a, \alpha)S = (b, \beta)S$ and hence for any $c \in Q$ there exists $(d, \gamma) \in S$ such that $(a, \alpha)(c, \alpha) = (b, \beta)(d, \gamma)$. But then $ac = bd$ which implies $aQ \subseteqq bQ$. By symmetry, we also have $bQ \subseteqq aQ$ so that $aQ = bQ$. Since Q is basic, we infer that $a = b$, which in turn implies that $(a, \alpha)\psi = (b, \beta)\psi$. Consequently $\sigma_S \subseteqq \ker \psi$. To prove the converse, by symmetry, it suffices to show that $(a, \alpha)S \subseteqq (a, \beta)S$ for any $a \in Q$, $\alpha, \beta \in \Phi$. Indeed, for any $(c, \gamma) \in S$, we obtain

$$(a, \alpha)(c, \gamma) = (ac, [a, c] + \alpha + \gamma) = (a, \beta)(c, \alpha + \gamma - \beta) \in (a, \beta)S.$$

Thus $\sigma_S = \ker \psi$ and $S/\sigma_S \cong Q$.

We prove next the second isomorphism. Fix $b \in Q$ and let 0 be the identity element of Φ. For any $\lambda \in GZ(S)$, $\lambda(b, 0)$ is of the form (b, β) for some $\beta \in \Phi$, for as we have proved above, $(a, \alpha)\sigma_S(b, \beta)$ if and only if $a = b$, i.e., λ must preserve the first entry. Hence we may define a function χ from $GZ(S)$ into Φ as follows:

$$\lambda(b, 0) = (b, \lambda\chi) \qquad (\lambda \in GZ(S)).$$

For $\lambda, \lambda' \in GZ(S)$, we have

$$(b, 0)[\lambda\lambda'(b, 0)] = \lambda(b, 0)\lambda'(b, 0) = (b, \lambda\chi)(b, \lambda'\chi) = (b^2, [b, b] + \lambda\chi + \lambda'\chi),$$

$$(b, 0)[\lambda\lambda'(b, 0)] = (b, 0)(b, (\lambda\lambda')\chi) = (b^2, [b, b] + (\lambda\lambda')\chi),$$

which implies $\lambda\chi+\lambda'\chi=(\lambda\lambda')\chi$, i.e., χ is a homomorphism. If $\lambda\chi=\lambda'\chi$, then $\lambda(b, 0)= =\lambda'(b, 0)$, and hence for any $(a, \alpha)\in S$,

$$[\lambda(a, \alpha)](b, 0)=[\lambda(b, 0)](a, \alpha)=[\lambda'(b, 0)](a, \alpha)=[\lambda(a, \alpha)](b, 0)$$

which shows that $\lambda=\lambda'$, so χ is one-to-one. Next let $\beta\in\Phi$. From what we have seen above, it follows that $(b, \beta)S=(b, 0)S$. It then follows easily that the function λ defined by the formula:

$$(b, \beta)(a, \alpha)=(b, 0)(\lambda(a, \alpha)) \qquad ((a, \alpha)\in S)$$

has the properties $\lambda\in GZ(S)$ and $\lambda(b, 0)=(b, \beta)$. Hence $\lambda\chi=\beta$ proving that χ maps $GZ(S)$ onto Φ. Therefore $GZ(S)\cong\Phi$.

The converse follows immediately from 4. 3 and 7. 5, formula (26) follows from commutativity of S.

7. 7 Corollary. *Let $S=(Q, \Phi, [\ ,\ ])$ and $S'=(Q', \Phi', [\ ,\ ]')$ where $[\ ,\ ]$ and $[\ ,\ ]'$ satisfy* (26), *Q and Q' are basic. Then the conditions in* 4. 4 *are necessary and sufficient for isomorphism of S and S'.*

Proof. This follows from 7. 6 and the proof of 4. 4.

7. 8 Corollary. *Every commutative cancellative semigroup S can be embedded into $\Gamma(Q)wl\,\Phi$ and l-densely and densely embedded into $\Lambda(Q)wl\,\Phi$ for some basic semigroup Q and an abelian group Φ.*

Proof. This follows from 7. 6 and 5. 4.

As an example, we compute all functions $[\ ,\ ]$ for two very simple basic semigroups.

7. 9 Proposition. *Let N be the additive semigroup of positive integers and Φ be an abelian group. For a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of elements of Φ, define a function $[\ ,\ ]$ by:*

$$[m, 1]=\alpha_m,\ [m, n]=\alpha_m+\sum_{i=1}^{n-1}(\alpha_{m+i}-\alpha_i) \quad (m\geqq 1, n\geqq 2). \tag{27}$$

Then $[\ ,\ ]$ satisfies both (6) *and* (26). *Conversely, every function $[\ ,\ ]$ from N into Φ satisfying* (6) *can be so obtained.*

Proof. For a function $[\ ,\ ]$ defined by (27), it is routine to verify that it satisfies (6) and (26). Conversely, let $[\ ,\ ]$ be a function from N into Φ satisfying (6). We let $\alpha_m=[m, 1]$ $(m\in N)$, then the second part of (27) can be written as

$$[m, n]=[m, 1]+\sum_{i=1}^{n-1}([m+i, 1]-[i, 1]) \quad (m\geqq 1, n\geqq 2).$$

The proof of this relation is by induction on n for a fixed m. The case of $n=1$ is trivial. Suppose the formula correct for n. By (6), we have

$$[m, n]+[m+n, 1] = [m, n+1]+[n, 1]$$

which implies

$$[m, n+1] = [m, n]+[m+n, 1]-[n, 1] =$$

$$= [m, 1]+\sum_{i=1}^{n-1}([m+i, 1]-[i, 1])+[m+n, 1]-[n, 1] =$$

$$= [m, 1]+\sum_{i=1}^{n}([m+i, 1]-[i, 1])$$

as required.

7. 10 Proposition. *Let N° be the additive semigroup of nonnegative integers and Φ be an abelian group. For any sequence $\{\alpha_n\}_{n=0}^{\infty}$ of elements of Φ define a function $[\ ,\ ]$ by: $[m, 0]=[0, m]=\alpha_0$ for $m\geqq 0$ and (27) for the remaining values. Then $[\ ,\ ]$ satisfies both (6) and (26). Conversely, every function $[\ ,\ ]$ from N° into Φ satisfying (6) can be so obtained.*

Proof. A proof using 7. 9 and considering the extra elements of the form $[m, 0]$, $[0, m]$ is straightforward and is omitted.

7. 11 Corollary. *If S is a cancellative semigroup such that S/σ_S is isomorphic to either N or N°, then S is commutative.*

Proof. By 4. 3, S is isomorphic to a Schreier extension of the abelian group $GZ(S)$ and N or N°. Now 7. 9 and 7. 10 imply that the corresponding function $[\ ,\ ]$ automatically satisfies (26) which implies the commutativity of the Schreier extension and thus also of S.

Since 7. 9 and 7. 10 yield all functions $[\ ,\ ]$, we are able to construct all Schreier extensions of Φ by N or N°. Even though 7. 7 gives necessary and sufficient conditions for isomorphism of such extensions, we are unable to tell which sequences will yield isomorphic semigroups.

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Isomorphism types of objects in categories determined by numbers of morphisms

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Consider a category $\mathfrak{A}$ and, for an object a of $\mathfrak{A}$ the family $(|\mathfrak{A}(x, a)|)_{x \in \mathrm{obj}\, \mathfrak{A}}$ of numbers of morphisms from all objects of $\mathfrak{A}$ into a. In [3], L. LOVÁSZ showed that the category of finite sets with systems of k-nary relations and the category of finite k-partite structures have the property that this family of numbers determines the isomorphism type of a, and used this fact to prove his product cancellation laws. Later on, in [2], he developed another method enabling him to prove the cancellation laws for general categories.

In the present paper a sufficient condition is given (Theorem 2. 2) on a category $\mathfrak{A}$ to have the property described above (cf. Definition 1. 7 below). Moreover, we prove that for every category such that there is only a finite number of morphisms between any two objects there is a full product preserving embedding into a category with that property (Theorems 2. 5 and 3. 3), which gives an information on the structure of product semigroups of isomorphism types (Corollary 3. 4). Also, a proof is given of one of LOVÁSZ' cancellation laws which is not an immediate consequence of the property above, for general categories, but based on the original idea from [3]. In fact, the proof of Theorem 2. 2 also exploits the idea from [3] in general categorial language.

§ 1. Preliminaries

1. 1. Definitions and notation. The category of sets (finite sets) and their mappings will be denoted by Set (Set f). If $\mathfrak{A}$ is a category, the symbol $\mathfrak{A}$ is also used for the natural functor $\mathfrak{A}^{op} \times \mathfrak{A} \to$ Set (i.e., $\mathfrak{A}(a, b)$ is the set of morphisms from a into b in $\mathfrak{A}$, $\mathfrak{A}(\varphi, \psi)(\alpha) = \psi\alpha\varphi$).

A morphism $\varepsilon: a \to b$ in $\mathfrak{A}$ is said to be a quotient in $\mathfrak{A}$, if 1_b is its image. The set of quotients from a into b is denoted by Quo (a, b). The set of monomorphisms and isomorphisms from a into b is denoted by Mono (a, b) and Iso (a, b), respectively.

If M is a set, $|M|$ is its cardinality.

If A is a small category and $\mathfrak{B}$ a category, $\mathfrak{B}^A$ designates the category of all functors $A \to \mathfrak{B}$ and their transformations.

1.2. Remark. Obviously, the quotients which are epimorphisms are exactly the well known extremal epimorphisms.

1.3. Definition. A category $\mathfrak{A}$ is said to be *quasifinite* if every $\mathfrak{A}(a, b)$ is finite. On the other hand, the expression *locally finite* will be preserved for categories with only a finite number of non-equivalent monomorphisms into each object (in accordance with the common use of "locally small" and "colocally small").

1.4. The following well known statements will be used without mentioning (they are either explicitely in textbooks or quite trivial to prove):

1) If $\varphi = \mu \cdot \varepsilon$ and μ is an image of φ, then ε is a quotient.

2) If $\varphi = \mu \cdot \varepsilon$ with a monomorphism μ and a quotient ε, then μ is an image of φ.

3) If $\mathfrak{A}$ has equalizers and ε is a quotient in $\mathfrak{A}$ then ε is an epimorphism.

4) A locally finite category with intersections has images.

5) If $\mathfrak{B}$ is complete (cocomplete, finitely complete, finitely cocomplete) then every $\mathfrak{B}^A$ is. The evaluation functors $\mathfrak{B}^A \to \mathfrak{B}$ preserve limits and colimits, consequently, monomorphisms and epimorphisms.

6) If A is finite and $\mathfrak{B}$ locally finite then $\mathfrak{B}^A$ is locally finite.

1.5. Lemma (coincides with Lemma 1 in [2]). *Let $\mathfrak{A}(a, a)$, $\mathfrak{A}(b, b)$ be finite, let $\mu: a \to b$, $\nu: b \to a$ be monomorphisms (epimorphisms, resp.). Then μ and ν are isomorphisms.*

Proof. There are integers $n \geqq 0$, $k > 0$ with $(\nu\mu)^{n+k} = (\nu\mu)^n$. Since $\nu\mu$ is a monomorphism, hence, $\nu((\mu\nu)^{k-1}\mu) = 1$. Thus, ν is a retraction and a monomorphism, hence an isomorphism.

1.6. Lemma. *Let $\mathfrak{A}$ be locally finite quasifinite, let x, a, b be its objects, T a system of objects of $\mathfrak{A}$ containing exactly one representant of each isomorphism class. Then there are only finitely many $\varphi: x \to d$ such that $d \in T$ and* $|\mathrm{Mono}\,(d, a)| \neq |\mathrm{Mono}\,(d, b)|$.

Proof. If $|\mathrm{Mono}\,(d, a)| \neq |\mathrm{Mono}\,(d, b)|$, at least one of them is not zero. Since no two $d, t \in T$ are isomorphic and $\mathfrak{A}$ is locally finite, there are only finitely many such d. Since $\mathfrak{A}$ is quasifinite, the statement follows.

1.7. Definition. A semifinite category $\mathfrak{A}$ is said to be *combinatorial* if $(\forall x\, |\mathfrak{A}(x, a)| = |\mathfrak{A}(x, b)|)$ implies that a is isomorphic to b.

§ 2. A sufficient condition for a category to be combinatorial. Consequences

2. 1. Theorem. *Let a, b be objects of $\mathfrak{A}$ such that every $\varphi: a \to b$ has an image. Let T be a system of objects of $\mathfrak{A}$ containing exactly one representant of each isomorphism class. Then*

$$|\mathfrak{A}(a, b)| = \sum_{t \in T} |\mathrm{Iso}\,(t, t)|^{-1} |\mathrm{Quo}\,(a, t)| \cdot |\mathrm{Mono}\,(t, b)|.$$

Proof. Define an equivalence relation e_t on $R_t = \mathrm{Quo}\,(a, t) \times \mathrm{Mono}\,(t, b)$ by $(\varepsilon, \mu) e_t (\varepsilon', \mu')$ iff $\mu\varepsilon = \mu'\varepsilon'$. Thus, $|\mathfrak{A}(a, b)| = \sum_T |R_t/e_t|$. On the other hand (see 1. 4. 2)) obviously

$$|R_t/e_t| = |\mathrm{Iso}\,(t, t)|^{-1} \cdot |\mathrm{Quo}\,(a, t)| \cdot |\mathrm{Mono}\,(t, b)|.$$

2. 2. Theorem. *Let $\mathfrak{A}$ be a locally finite quasifinite category with images such that every quotient is an epimorphism. Then it is combinatorial.*

Proof. Let $|\mathfrak{A}(t, a)| = |\mathfrak{A}(t, b)|$ for every $t \in T$ (T from 2. 1). We shall prove that then $|\mathrm{Mono}\,(t, a)| = |\mathrm{Mono}\,(t, b)|$ for every $t \in T$ which shall prove the statement by 1. 5. Thus, let $|\mathrm{Mono}\,(t, a)| \neq |\mathrm{Mono}\,(t, b)|$. By 2. 1,

$$0 = \sum_{d \neq t} \frac{|\mathrm{Quo}\,(t, d)|}{|\mathrm{Iso}\,(d, d)|} \cdot \big(|\mathrm{Mono}\,(d, a)| - |\mathrm{Mono}\,(d, b)|\big) + \big(|\mathrm{Mono}\,(t, a)| - |\mathrm{Mono}\,(t, b)|\big).$$

($\mathrm{Quo}\,(t, t) = \mathrm{Iso}\,(t, t)$ by 1. 5 and the assumption on quotients). Thus, there is a non-isomorphic quotient $\varepsilon(t): t \to \bar{t}$ such that $|\mathrm{Mono}\,(\bar{t}, a)| \neq |\mathrm{Mono}\,(\bar{t}, b)|$. Put $t_0 = t$, $t_{n+1} = \bar{t}_n$, $\varepsilon_n = \varepsilon(t_n)$. By 1. 6 there are natural k and $n > k$ with $\varepsilon_k \cdot \varepsilon_{k-1} \cdot \dots \cdot \varepsilon_0 = \varepsilon_n \cdot \varepsilon_{n-1} \cdot \dots \cdot \varepsilon_0$. By the assumption on quotients, thus, $\varepsilon_n \cdot \dots \cdot \varepsilon_{k+1} = 1$. Hence ε_{k+1} is a coretraction and an epimorphism, so that it is an isomorphism, which is a contradiction.

2. 3. Corollary. *Every locally finite quasifinite finitely complete category is combinatorial.*

Proof. Since it is locally finite and has finite intersections, it has images (see e.g. [4]). Since it has equalizers, the quotients are epimorphisms.

2. 4. Remark. Theorem 2. 2 holds also under dual conditions. It suffices to use an analogon of 2. 1 with coimage decomposition of the morphisms, epimorphisms instead of quotients and subobjects (i.e. morphisms with the identity for a coimage) instead of monomorphisms.

2. 5. Theorem. *For every finite category A there is a full product preserving embedding $\Phi: A \to C$ with C combinatorial.*

Proof. The Yoneda embedding $Y:A\to\mathrm{Setf}^{A^{op}}$ (given by $Y(a)(b)=A(b,a)$, $Y(\varphi)^b=A(\varphi,1_b)$) is a full product preserving embedding. By 2. 3 and 1. 4. 5) 6), $\mathrm{Setf}^{A^{op}}$ is combinatorial.

2. 6. As an immediate consequence of 2. 5 and of the trivial fact that $|\mathfrak{A}(x,a\times b)| = |\mathfrak{A}(x\cdot a)|\cdot|\mathfrak{A}(x,b)|$ we obtain (exactly like the analogous statements for special categories in [3]) the general LovÁsz' cancellation laws from [2] (we reformulate them slightly):

1) Let $n\geqq 1$ and a^n be isomorphic to b^n in a category $\mathfrak{A}$. If there is only a finite number of morphisms between a, b, a^n then a is isomorphic to b.

2) Let $a\times c$ be isomorphic to $b\times c$ in $\mathfrak{A}$, let $\mathfrak{A}(d,c)$ be non-void. If there is only a finite number of morphisms between a, b, c, $a\times c$, $a\times d$, $b\times d$ then $a\times d$ is isomorphic to $b\times d$.

By duality, analogous statements hold for copowers and sums.

2. 7. Now, we shall show that also the one cancellation law which is not an immediate consequence of combinatoriality (Theorem 4 in [2]) can be proved in a way analogous to that of Theorem 5 in [3].

Theorem. *Let a, b, c, d be objects of $\mathfrak{A}$ such that there is only a finite number of morphisms between them, let*

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & a \\
d \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & \\
 & c
\end{tikzcd}
,\qquad
\begin{tikzcd}[row sep=small, column sep=small]
 & b \\
d \arrow[ur, "\beta"] \arrow[dr, "\gamma'"'] & \\
 & c
\end{tikzcd}
$$

be products. Then there is an isomorphism $\delta:d\to d$ such that $\gamma'\cdot\delta=\gamma$.

Proof. Let A be the full subcategory of $\mathfrak{A}$ generated by a,b,c,d, $Y:A\to\mathrm{Setf}^{A^{op}}$ the Yoneda embedding (see 2. 5), denote by

$$k:Y(d)\to Y(a)\times Y(c),\quad k':Y(d)\to Y(b)\times Y(c)$$

the natural equivalences defined by $k^x(\mu)=(\alpha\mu,\gamma\mu)$, $k'^x(\mu)=(\beta\mu,\gamma'\mu)$. Let B be the category the objects of which are couples (x,ξ) with x an object of A, $\xi:x\to c$, the morphisms from (x,ξ) into (y,η) being the triples (ξ,φ,η) with $\varphi:y\to x$ such that $\xi\varphi=\eta$. $U:B\to A^{op}$ is defined by $U(x,\xi)=x$, $U(\xi,\varphi,\eta)=\varphi$, $U^*:\mathrm{Setf}^{A^{op}}\to\mathrm{Setf}^B$ by $U^*(f)=f\circ U$, $U^*(\tau)=\tau U$. U^* obviously preserves products (see 1. 4. 5)).

Finally, define a functor $L:\mathrm{Setf}^B\to\mathrm{Setf}^{A^{op}}$ by

$$L(f)(x)=U\{f(x,\xi)\times\{\xi\}|\xi:x\to c\},\quad L(f)(\varphi)(u,\xi)=\bigl(f(\xi,\varphi,\xi\varphi)(u),\xi\varphi\bigr),$$

and

$$L(\tau)^x(u,\xi)=\bigl(\tau^{(x,\xi)}(u),\xi\bigr)\quad\text{for}\quad\tau:f\to g.$$

Let $e:B\to\mathrm{Set}f$ be the constant functor defined by $e(\alpha)=1_{\{0\}}$. It is a singleton of $\mathrm{Set}f^B$ and we have a transformation $\tau:e\to U^*Y(c)$, namely that defined by $\tau^{(x,\xi)}(0)=\xi$.

Thus, since Setf^B is combinatorial and since U^* preserves products, we have by 2. 6. 2) an isomorphism $\vartheta: U^* Y(a) \to U^* Y(b)$. Now, consider $k'^{-1} \cdot L(\vartheta) \cdot k: Y(d) \to \to Y(d)$. Since Y is full, it is equal to some $Y(\delta)$ with $\delta: d \to d$. We have

$$\gamma' \delta = p_2 k'^d(\delta) = p_2\big(L(\vartheta)^d(\alpha, \gamma)\big) = p_2\big(\vartheta^{(d,\gamma)}(\alpha), \gamma\big) = \gamma$$

(p_2 is the natural projection $A(d, b) \times A(d, c) \to A(d, c)$).

§ 3. Embedding of infinite categories into combinatorial ones

3. 1. In this paragraph we shall use the following known statements:

1) Let A, B, C be categories, A small and C cocomplete (or A finite, B quasi-finite and C finitely cocomplete). Let $F: A \to B$ be a functor, let $F^*: C^B \to C^A$ be defined by $F^*(f) = f \circ F$, $F^*(\tau) = \tau F$. Then F^* has a left adjoint.

2) If F is a full embedding and L a left adjoint to F^*, then $F^* \circ L \cong 1$.

3) Let C be Set or Setf, $Y_1: A^{op} \to C^A$, $Y_2: B^{op} \to C^B$ the Yoneda embeddings, $F: A \to B$ arbitrary, $F': A^{op} \to B^{op}$ given by the same formula as F, L a left adjoint to F^*. Then $L \circ Y_1 \cong Y_2 \circ F'$.

The first one is, in substance, due to Kan and a proof can be found in [5], p. 74, the second to Freyd. The third is due to Isbell ([1]).

3. 2. Remark. The functor L from 2. 7 is the left adjoint to U^* granted by 3. 1. 1).

3. 3. Theorem. *For every quasifinite category $\mathfrak{A}$ there is a full product preserving embedding into a combinatorial category $\mathfrak{B}$. If $\mathfrak{A}$ is finite, $\mathfrak{B}$ can be found countable, if $\mathfrak{A}$ is small infinite, $\mathfrak{B}$ can be found with $|\mathfrak{A}| = |\mathfrak{B}|$.*

Proof. For a full finite subcategory A of $\mathfrak{A}$ denote by J_A the embedding $A \subset \mathfrak{A}$, by L_A a left adjoint to J_A^*: Setf$^{\mathfrak{A}^{op}} \to$ Setf$^{A^{op}}$.

Denote by $\mathfrak{B}$ the full subcategory of Setf$^{\mathfrak{A}^{op}}$ generated by the functors isomorphic to those of the form $L_A(f)$. By 3. 1. 3), the Yoneda embedding $Y: \mathfrak{A} \to$ Setf$^{\mathfrak{A}^{op}}$ maps $\mathfrak{A}$ into $\mathfrak{B}$. Thus, a full product preserving embedding is obtained. Now, it suffices to show that $\mathfrak{B}$ is combinatorial. We have $\mathfrak{B}(L_A f, g) \cong$ Setf$^A(f, J_A^* g)$ always finite. Let $g_i = L_{A_i}(h_i)$, $i = 1, 2$, be such that, for every g, $|\mathfrak{B}(g, g_1)| = |\mathfrak{B}(g, g_2)|$. Denote by A the full subcategory of $\mathfrak{A}$ generated by A_1 and A_2, by J_i the embedding $A_i \subset A$, by L_i a left adjoint to J_i^*. Obviously $L_{A_i} \cong L_A \circ L_i$ and hence $g_i \cong L_A(h_i')$ where $h_i' = L_i(h_i)$. For any $f: A \to$ Setf, by 3. 1. 2),

$$|\mathrm{Setf}^{A^{op}}(f, h_i')| = |\mathrm{Setf}^{A^{op}}(f, J_A^* L_A(h_i'))| = |\mathfrak{B}(L_A f, g_i)|.$$

Thus, since Setf$^{A^{op}}$ is combinatorial, we have $h_1' \cong h_2'$ and hence $g_1 \cong g_2$.

3.4. Corollary. *Let $\mathfrak{A}$ be a quasifinite category, let S be the partial binary algebra the elements of which are isomorphism types of objects of $\mathfrak{A}$, with product (sum, resp.) as the partial operation. Denote by N the multiplicative semigroup of natural numbers. Then, for M suitably large, S is isomorphic to a subalgebra of N^M.*

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A note on small categories with zero

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Let S be a semigroup with zero 0. We say that an idempotent e is a categorical left unit if ex is either x or 0 for any $x \in S$. An idempotent e is a categorical right unit if xe is either x or 0 for any $x \in S$.

A semigroup with zero is called *categorical at zero* if $abc=0$ implies either $ab=0$ or $bc=0$.

In accordance with [1] we define:

Definition. A semigroup with 0 is called a *small category with zero* if it satisfies the following two conditions:

C_1. To any non-zero element $a \in S$ there is a categorical left unit $e_l(a)$ and a categorical right unit $e_r(a)$ such that $e_l(a) \cdot a = a \cdot e_r(a) = a$.

C_2. S is categorical at zero.

For brevity we shall denote in the following a small category with zero as a C-semigroup.

The connection between category theory and C-semigroups is well known. C-semigroups have been studied by several authors, in particular by HOEHNKE (see e.g. [2]).

The purpose of this note is to prove a theorem concerning the relation between C-semigroups and a class of semigroups called dual semigroups introduced by the author ([3], [4]).

1

The following is partly presented in [1] in form of Exercises:

Lemma 1. *In a semigroup satisfying Condition* C_1 *we have:*

a) $e_r(a)$, $e_l(a)$ *are uniquely determined.*

b) $a \in Sa$, $a \in aS$, *in particular* $S^2 = S$.

c) *Any categorical left unit of* S *is a categorical right unit of* S.

With respect to c) we may speak (in a semigroup satisfying C_1) about the set of all non-zero categorical idempotents $\in S$. This set will be denoted by E. E is a subset of the set of all non-zero idempotents E_0 and simple examples show that E can be a proper subset of E_0.

Note that if card $E=1$, i.e. $E=\{e\}$, then e is the two-sided unit element of S.

Let $e_1, e_2 \in E$ and $e_1 \neq e_2$. Then $Se_1 \cap Se_2 = 0$. For if there were $0 \neq a \in Se_1 \cap \cap Se_2$ we would have $a=be_1=ce_2$ $(b, c \in S)$, hence $a=ae_1=ae_2$, a contradiction to a). Since any $a \in S$ can be written in the form $a=ae$, $e \in E$, we have

Lemma 2. *Any semigroup satisfying Condition* C_1 *can be written in the form*

$$S=\bigcup_{e_\alpha \in E} Se_\alpha=\bigcup_{e_\alpha \in E} e_\alpha S,$$

where $Se_\alpha \cap Se_\beta = e_\alpha S \cap e_\beta S = 0$ *for* $e_\alpha \neq e_\beta$.

Lemma 3. *If* $e_1, e_2 \in E$ *and* $e_1 \neq e_2$, *then* $e_1 e_2=0$.

Proof. Suppose that $e_1 e_2 \neq 0$. Then since both e_1 and e_2 are categorical units we have $e_1 e_2=e_1$ and $e_1 e_2=e_2$, hence $e_1=e_2$, contrary to the assumption.

We now use Condition C_2.

Lemma 4. *If* S *is a C-semigroup,* $e \in E$ *and* $0 \neq a \in Se$, $0 \neq b \in eS$, *then* $ab \neq 0$.

Proof. $ab=aeb=0$ implies either $ae=0$ or $eb=0$, i.e. either $a=0$ or $b=0$.

Remark. There are large classes of semigroups satisfying Condition C_2. E.g., a 0-simple semigroup containing a 0-minimal left ideal (in particular a completely 0-simple semigroup) is categorical at zero. (For a proof see [1], Lemma 8, 23, p. 86.)

2

In [3] (see also [1], pp. 29—30) we have introduced the notion of a dual semigroup. If S is a semigroup with zero and A a subset of S, the *left* and *right annihilators* $r(A)$ and $l(A)$ are defined by $r(A)=\{x \in S | Ax=0\}$, $l(A)=\{x \in S | xA=0\}$.

A semigroup with zero is called *dual* if for any left ideal L of S we have $l[r(L)]=L$ and for any right ideal R of S we have $r[l(R)]=R$.

Clearly $l(A)$ and $r(A)$ are left and right ideals, respectively.

It has been proved in [4] that in a dual semigroup S any left ideal of S contains a 0-minimal left ideal of S. If L_0 is a 0-minimal left ideal of S, then $r(L_0)$ is a maximal right ideal of S. Analogously for right ideals. Also if for two left ideals we have $L_1 \subsetneqq L_2$, then $r(L_1) \supsetneqq r(L_2)$.

Recall for further purposes: A semigroup S is called a 0-*direct union* of its two-sided ideals M_α, $\alpha \in A$, if

$$S=\bigcup_{\alpha \in A} M_\alpha \quad \text{and} \quad M_\alpha M_\beta = M_\alpha \cap M_\beta = 0 \quad \text{for} \quad \alpha \neq \beta.$$

We now prove the following

Theorem. *A small category with zero is a dual semigroup if and only if it is a 0-direct union of completely 0-simple dual semigroups.*

Proof. 1. Let S be a C-semigroup. Consider the left ideal Se, $e \in E$. Since $Se \cdot fS = 0$ for every $f \neq e$ $(e, f \in E)$, we have $r(Se) \supset \{\bigcup_f fS \mid f \in E, f \neq e\}$. By Lemma 4 for any $a \in eS$ we have $Se \cdot a \neq 0$. Hence $r(Se) = \{\bigcup_f fS \mid f \in E, f \neq e\}$.

We next prove: If S is moreover dual, then Se is a 0-minimal left ideal of S. Let $L \neq 0$ be a left ideal of S, $L \subsetneq Se$. By duality we have $r(Se) \subsetneq r(L)$. Hence $r(L) \supsetneq$ $\supsetneq \{\bigcup fS \mid f \in E, f \neq e\}$ and $r(L) \cap eS \neq 0$. There exists therefore an element $a \in eS$ such that $La = 0$. This constitutes a contradiction since for any $b \in L$, $b \neq 0$ and $a \in eS$, $a \neq 0$ we have (by Lemma 4) $ba \neq 0$. Hence Se is a 0-minimal left ideal of S.

Call two ideals $A_1 \neq A_2$ of S *quasidisjoint* if $A_1 \cap A_2 = 0$.

Lemma 2 implies that S is a union of pairwise quasidisjoint 0-minimal left ideals of S. Analogously, S is a union of pairwise quasidisjoint 0-minimal right ideals of S. It is known (see e.g. [1], Theorem 6, 39) that such a semigroup is a 0-direct union of completely 0-simple semigroups. Hence we may write

$$S = \bigcup_{\alpha \in A} M_\alpha, \quad M_\alpha M_\beta = M_\alpha \cap M_\beta = 0 \quad \text{for} \quad \alpha \neq \beta,$$

where M_α $(\alpha \in A)$ runs through all 0-minimal two-sided ideals of S.

Now it is easy to see that a semigroup which is a 0-direct union of its ideals is dual iff each of the ideal components is dual. (For an explicit proof see [4].) Hence each M_α $(\alpha \in A)$ is dual.

2. Suppose conversely that S is a 0-direct union of completely 0-simple dual semigroups: $S = \bigcup_{\alpha \in A} M_\alpha$, $M_\alpha M_\beta = M_\alpha \cap M_\beta = 0$ for $\alpha \neq \beta$.

Recall (see [4]) that a completely 0-simple dual semigroup M_α can be written in the form $M_\alpha = \bigcup_{e = e^2 \in M_\alpha} M_\alpha e = \bigcup_{e = e^2 \in M_\alpha} eM_\alpha$ with $M_\alpha e_1 \cap M_\alpha e_2 = 0$ and $e_1 M_\alpha \cap$ $\cap e_2 M_\alpha = 0$ for $e_1 \neq e_2$, and any non-zero idempotent $\in M_\alpha$ is a categorical unit of M_α. This immediately implies that the 0-direct union $S = \bigcup_{\alpha \in A} M_\alpha$ satisfies Condition C_1. Since (by the Remark after Lemma 4 above) each M_α is categorical at zero, S is clearly also categorical at zero. Hence S satisfies Condition C_2. This proves our Theorem.

Remark 1. In [4] we have proved that any dual semigroup satisfies Condition C_1 and, moreover, in any dual semigroup every non-zero idempotent is a categorical unit. This implies:

Corollary 1. *A dual semigroup which is not a 0-direct union of completely 0-simple dual semigroups is not categorical at zero.*

Remark 2. Recall the following notions. An element $a \neq 0$ of a C-semigroup S is called *invertible*, if there is an element $a' \in S$ such that $aa' = e_l(a)$, $a'a = e_r(a)$. Further, a regular semigroup is said to be primitive if each of its non-zero idempotents is primitive. Using the known results mentioned in [1] (p. 79) and the fact proved in [5] (see also [4]) that any 0-simple dual semigroup is completely 0-simple, we have the following

Corollary 2. *For a semigroup with zero the following statements are equivalent:*

1. *S is a small category with zero which is a dual semigroup.*
2. *S is a 0-direct union of 0-simple dual semigroups.*
3. *S is a small category with zero every non-zero element of which is invertible.*
4. *S is a primitive inverse semigroup.*

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Hausdorff operators

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If $x=\{x_1, x_2, x_3, \dots\}$ is any complex sequence and if C_0 is the Cesàro matrix then $(C_0 x)_n$, the nth term of the transformed sequence, is given by $\frac{1}{n}\sum_{k=1}^{n} x_k$. The Cesàro matrix belongs to a class of triangular matrices which are known as Hausdorff matrices In [1] A. BROWN, P. R. HALMOS, and A. L. SHIELDS have shown that the Cesàro matrix defines a bounded operator on l^2, whose spectrum is $\{z: |z-1| \leqq 1\}$ and that it is hyponormal. Whether or not this operator is subnormal is left by them as an open question. T. L. KRIETE, III, and DAVID TRUTT in [3] show that the Cesàro operator on l^2 is subnormal. We show that certain more general Hausdorff matrices as operators on l^2 are also subnormal.

In [1] the continuous analogues of the Cesàro matrix are also studied. In particular, it is shown that if C and C_1 denote the continuous analogues of C_0 on $L^2(0, 1)$ and on $L^2(0, \infty)$, respectively, then $I-C^*$ is a simple unilateral shift and $I-C_1^*$ is a simple bilateral shift. We show that $I-\Gamma_a^{1*}/N$, in the notation of [4], as an operator on $L^2(0, 1)$ (or $L^2(0, \infty)$) is a simple unilateral shift (or a simple bilateral shift) for $a>1/2$. I wish to thank Professors B. E. RHOADES and J. P. WILLIAMS for many fruitful conversations. I am grateful to the referee for his comments which resulted in simplifying the proofs of Theorems 3 and 4.

Theorem 1. *Let $A=(a_{nk})$ be a real triangular matrix which is a bounded operator on l^2. Then $\sigma(A)$, the spectrum of A, is symmetric about the real axis.*

Proof. Let $\lambda = a+ib$ be any complex number such that $b\neq 0$. Then

$$A-\lambda I = (A-aI)-ibI = b\left[\frac{1}{b}(A-aI)-iI\right].$$

Since A is real, $\frac{1}{b}(A-aI)$ is real. Therefore, it suffices to show that i is not in $\sigma(A)$ if and only $-i$ is not. Since $A-iI$ and $A+iI$ are clearly one-one, it suffices to show that both are simultaneously onto. But if x, y, u, v are real sequences in l^2 then

$(A+iI)(x+iy) = u+iv$ implies $(A-iI)(x-iy) = u-iv$ and vice versa. Hence the theorem follows.

Corollary. *Let H be a Hausdorff matrix with real entries which is a bounded operator on l^2. Then $\sigma(H)$ is symmetric about the real axis.*

Theorem 2. *Every Hausdorff matrix H that is a bounded operator on l^2 is subnormal.*

Proof. We know from [3] that $I-C_0$ is subnormal with a cyclic vector. It follows from [2] that H commutes with $I-C_0$. The theorem now follows from [5].

Definition 1. For each $a>1/2$ let B_a be the operator defined on $L^2(0, \infty)$ by

$$B_a f(x) = f(x)-(2a-1)x^{a-1}\int_x^\infty f(s)s^{-a}\,ds.$$

Note that $B_1 = I-C_1^*$ is the operator studied in [1]. It follows from [4] that B_a is a bounded operator.

Theorem 3. *B_a is a simple bilateral shift on $L^2(0, \infty)$.*

Proof. Consider the map $T_a: L^2(0, \infty)\to L^2(0, \infty)$ such that for any f in $L^2(0, \infty)$

$$T_a f(x) = \sqrt{2a-1}\,x^{a-1}f(x^{2a-1}).$$

A change of variable argument shows that T_a is norm preserving. It is easy to check that $T_a^{-1}=T_b$ where $b = \dfrac{a}{2a-1}$. Hence T_a is unitary. Observe also that for any f in $L^2(0, \infty)$

$$T_a B_1 f = B_a T_a f.$$

Hence B_a is unitarily equivalent to B_1. Since B_1 is a simple bilateral shift [1, Theorem 5], it follows that B_a is also a simple bilateral shift.

Since the spectrum of a simple bilateral shift is the unit circle, Theorem 3 yields the following result of Rhoades [4, Theorem 16]:

Corollary. *$\sigma(\Gamma_a^{1*}) = \{z: |z-N| = N\}$, where $N = \dfrac{2a}{2a-1}$ and $B_a = I-\Gamma_a^{1*}/N$.*

Remark. $\{T_a : a>1/2\}$ forms a unitary group under the composition $T_aT_b = T_{a\oplus b}$ where $a\oplus b = 2ab-a-b+1$ for $a, b>1/2$. Observe that the set of real numbers greater than $1/2$ forms a group under the composition $\oplus$ and that this group is isomorphic to the group of positive real numbers under the map $\alpha\to 2\alpha-1$.

Definition 2. For each $a>1/2$ let V_a be the operator defined on $L^2(0, 1)$ by

$$V_a f(x) = f(x)-(2a-1)x^{a-1}\int_x^1 f(s)s^{-a}\,ds.$$

Note that $V_1 = I-C^*$ is the operator studied in [1].

Theorem 4. *V_a is a simple unilateral shift.*

Proof. Since $L^2(0,1)$ is invariant under T_a, T_a^{-1}, B_a, and B_1, and since, as we have seen in the proof of Theorem 3, $T_aB_1=B_aT_a$ we get $T_a|_{L^2(0,1)}V_1=V_aT_a|_{L^2(0,1)}$ because $B_1|_{L^2(0,1)}=V_1$ and $B_a|_{L^2(0,1)}=V_a$. Observe that $T_a|_{L^2(0,1)}$ is also unitary to conclude that V_a is unitarily equivalent to V_1. Since V_1 is a simple unilateral shift [1, Theorem 4], the theorem follows.

Since the spectrum of a simple unilateral shift is the unit disk, Theorem 4 yields the following result of RHOADES [4, Theorem 10]:

Corollary. *$\sigma(\Gamma_a^{1*})=\{z\colon |z-N|\leqq N\}$, where $N=\dfrac{2a}{2a-1}$ and $V_a=I-\Gamma_a^{1*}/N$.*

Remark. The orthogonal complement of the range of V_1 is the one-dimensional subspace spanned by the constant function $e(x)\equiv 1$. Since V_1 is a simple unilateral shift it follows that $\{e, V_1e, V_1^2e, \dots\}$ is an orthonormal basis for $L^2(0,1)$. Since

$$V_1^n e(x) = 1+\binom{n}{1}\log x+\binom{n}{2}\frac{1}{2!}(\log x)^2+\dots+\frac{1}{n!}(\log x)^n \quad \text{for} \quad n\geqq 1,$$

it follows that *polynomials in* $\log x$ *are dense in* $L^2(0,1)$. Observe that the map $S\colon L^2(0,1)\to L^2(1,\infty)$ given by $Sf(x)=\dfrac{1}{x}f\left(\dfrac{1}{x}\right)$ is unitary. Since $S(\log x)^n=$ $=\dfrac{(-1)^n}{x}(\log x)^n$, it follows that *the linear span of* $\left\{\dfrac{1}{x}, \dfrac{\log x}{x}, \dfrac{(\log x)^2}{x}, \dots\right\}$ *is dense in* $L^2(1,\infty)$.

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Eine Abschätzung des Maximums der Partialsummen von Orthogonalreihen

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1. Es sei $\lambda(x)$ eine monoton wachsende Funktion mit $\lambda(1) \geqq 1$, $\lim_{x\to\infty} \lambda(x) = \infty$ und $\lambda(x) = O(\log^2 x)$. Wir bezeichnen mit Λ^* die Klasse der im Intervall $(0, 1)$ orthonormierten Systeme $\varphi = \{\varphi_n(x)\}_1^\infty$, für die

$$L_n(\varphi; x) = \int_0^1 \left| \sum_{k=1}^n \varphi_k(x)\varphi_k(t) \right| dt \leqq \lambda(n) \qquad (0 \leqq x \leqq 1;\ n = 1, 2, \ldots)$$

erfüllt ist. Für eine Folge $a = \{a_n\}_1^\infty$ setzen wir

$$\|a; \lambda\|^* = \sup_{\varphi \in \Lambda^*} \int_0^1 \sup_{1 \leqq i \leqq j} |a_i\varphi_i(x) + \cdots + a_j\varphi_j(x)|\, dx.$$

In dieser Note werden wir die folgenden Behauptungen beweisen. ($C_1, C_2, \ldots$ bezeichnen im Folgenden positive Konstanten.)

Satz I. *Es sei $\lambda(x)$ konkav von unten. Ist $|a_n| \geqq |a_{n+1}|$ $(n = 1, 2, \ldots)$, dann gilt*

$$C_1 \left\{ \sum_{n=1}^\infty a_n^2 \lambda(n) \right\}^{1/2} \leqq \|a; \lambda\|^*. \tag{1}$$

Satz II. *Es sei $\lambda(x)$ konkav von unten mit $\lambda(n^2) \leqq C_2 \lambda(n)$ $(n = 1, 2, \ldots)$. Dann gilt*

$$C_3 \left\{ \sum_{n=1}^\infty a_n^2 \lambda\left(\sum_{k=1}^\infty a_k^2 / a_n^2 \right) \right\}^{1/2} \leqq \|a; \lambda\|^*. \tag{2}$$ [1]

Ähnliche Abschätzungen wurden für die Klasse Λ^* aller in $(0, 1)$ orthonormierten Systeme, bzw. für die Klasse der in $(0, 1)$ orthonormierten Systeme φ mit der Eigenschaft $|\varphi_n(x)| \leqq K$ $(K > 1;\ 0 \leqq x \leqq 1;\ n = 1, 2, \ldots)$ bewiesen ([2], [3], [4], [5]).

[1] Im Falle $a_k = 0$ soll man unter $a_n^2 \lambda\left(\sum_{k=1}^\infty a_k^2/a_n^2 \right)$ 0 verstehen.

Es sei Λ die Klasse der im Intervall (0, 1) orthonormierten Systeme φ, für die

$$\int_0^1 \sup_n \frac{L_n(\varphi; x)}{\lambda(n)}\,dx \leqq 1$$

erfüllt ist. Für eine Folge a setzen wir

$$\|a; \lambda\| = \sup_{\varphi\in\Lambda} \int_0^1 \sup_{1\leqq i\leqq j} |a_i\varphi_i(x) + \ldots + a_j\varphi_j(x)|\,dx.$$

Offensichtlich gilt für jede Folge a

(3) $$\|a; \lambda\|^* \leqq \|a; \lambda\|.$$

Also ist Satz I die Verschärfung eines vorigen Resultates vom Verfasser ([4], Satz IV).

In [4] haben wir gezeigt, daß für jede Folge a

$$\|a; \lambda\| \leqq C_4 \left\{\sum_{n=1}^{\infty} a_n^2 \lambda(n)\right\}^{1/2}$$

besteht. Daraus und aus (3) folgt, auf Grund des Satzes I, daß im Falle, wenn $\lambda(x)$ konkav ist und $|a_n| \geqq |a_{n+1}|$ $(n=1, 2, \ldots)$ besteht, man $\|a; \lambda\|$ und $\|a; \lambda\|^*$ genau abschätzen kann:

$$C_1 \left\{\sum_{n=1}^{\infty} a_n^2 \lambda(n)\right\}^{1/2} \leqq \|a; \lambda\|, \quad \|a; \lambda\|^* \leqq C_4 \left\{\sum_{n=1}^{\infty} a_n^2 \lambda(n)\right\}^{1/2}.$$

Ist $\lambda(x)$ konkav mit $\lambda(n^2) \leqq C_2 \lambda(n)$ $(n=1, 2, \ldots)$, und gilt $|a_n| \geqq |a_{n+1}|$ $(n=1, 2, \ldots)$, dann sind die Abschätzungen (1) und (2) gleichwertig. Das folgt aus den Ungleichungen

(4) $$C_5 \sum_{n=1}^{\infty} a_n^2 \lambda\left(\sum_{k=1}^{\infty} a_k^2/a_n^2\right) \leqq \sum_{n=1}^{\infty} a_n^2 \lambda(n) \leqq \sum_{n=1}^{\infty} a_n^2 \lambda\left(\sum_{k=1}^{\infty} a_k^2/a_n^2\right).$$

Zum Beweis von (4) kann man $\sum_{n=1}^{\infty} a_n^2 = 1$ voraussetzen. Dann ist aber $na_n^2 \leqq 1$ $(n=1, 2, \ldots)$, und so gilt

$$\sum_{n=1}^{\infty} a_n^2 \lambda(n) \leqq \sum_{n=1}^{\infty} a_n^2 \lambda(1/a_n^2)$$

offensichtlich. Es seien σ_1 und σ_2 die Mengen derjenigen Indizes n, für die $a_n^2 \geqq 1/n^2$, bzw. $a_n^2 < 1/n^2$ erfüllt ist. Durch einfache Rechnung erhalten wir

$$\sum_{n=1}^{\infty} a_n^2 \lambda(1/a_n^2) = \sum_{n\in\sigma_1} a_n^2 \lambda(1/a_n^2) + \sum_{n\in\sigma_2} a_n^2 \lambda(1/a_n^2) \leqq$$

$$\leqq \sum_{n\in\sigma_1} a_n^2 \lambda(n^2) + C_6 \sum_{n\in\sigma_2} \frac{1}{n^{3/2}} a_n^{1/2} \log^2 (1/a_n^2) \leqq$$

$$\leqq C_7 \sum_{n=1}^{\infty} a_n^2 \lambda(n) + C_8 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \leqq C_9 \sum_{n=1}^{\infty} a_n^2 \lambda(n).$$

Damit haben wir (4) bewiesen.

In [6] haben wir gezeigt, daß im Falle $\|a;\lambda\|=\infty$ ein System $\varphi\in\Lambda$ derart existiert, daß die Reihe

$$\sum a_n\varphi_n(x) \tag{5}$$

in (0, 1) fast überall divergiert.

Auf Grund von (3) bekommen wir aus Satz I, bzw. aus Satz II:

Folgerung I. *Es sei $\lambda(x)$ konkav von unten. Gelten $|a_n|\geqq|a_{n+1}|$ $(n=1,2,\dots)$ und*

$$\sum a_n^2\lambda(n)=\infty,$$

dann gibt es $\varphi\in\Lambda$ derart, daß die Reihe (2) *in* (0, 1) *fast überall divergiert.*

Folgerung II. *Es sei $\lambda(x)$ konkav von unten, mit $\lambda(n^2)\leqq C_2\lambda(n)$ $(n=1,2,\dots)$. Gilt*

$$\sum a_n^2\lambda\left(\sum_{k=1}^{\infty}a_k^2/a_n^2\right)=\infty,$$

so gibt es ein System $\varphi\in\Lambda$ derart, daß die Reihe (5) *in* (0, 1) *fast überall divergiert.*

Bemerkung. Durch Anwendung der Methode in [4] kann man zeigen, daß Folgerungen I—II mit einem solchen System φ richtig sind, für welches

$$L_n(\varphi;x)=O(\lambda(n))\qquad(0\leqq x\leqq 1)$$

besteht.

2. Zum Beweis der Sätze werden wir einige Hilfssätze benützen.

Hilfssatz I. ([1], S. 46—50.) *Für das Haarsche System $\chi=\{\chi_n(x)\}$ gilt*

$$L_n(\chi;x)\leqq 1\qquad(0\leqq x\leqq 1;\ n=1,2,\dots).$$

Hilfssatz II. ([5], Hilfssatz XIV.) *Es seien $p(\geqq 2)$, q natürliche Zahlen. Dann gibt es ein in* (0, 1) *orthonormiertes System von Treppenfunktionen $g_l(p,q;x)$ $(l=1,\dots,2pq)$ mit den folgenden Eigenschaften. Es gilt*

$$\int_0^1\left|\sum_{l=1}^{n}g_l(p,q;x)g_l(p,q;t)\right|dt\leqq C_{10}\log^2 p\qquad(0\leqq x\leqq 1;\,n=1,\dots,2pq;\ C_{10}\geqq 1),$$

und es gibt eine einfache Menge $E(\subseteqq(0,1))$ mit $m(E)=\frac{1}{5}$ derart, daß für jedes $x\in E$ ein Index $m(x)(<2pq)$ existiert, mit $g_l(p,q;x)\geqq 0$ $(l=1,\dots,m(x))$ und

$$\sum_{l=1}^{m(x)}g_l(p,q;x)\geqq C_{11}\sqrt{2pq}\,\log p\,.$$

(Eine Funktion $f(x)$ in einem Intervall I wird eine Treppenfunktion genannt, wenn sie stückweise konstant ist. Eine Menge wird einfach genannt, wenn sie die Vereinigung endlichvieler Intervalle ist. In dieser Note bezeichnet $\log\alpha$ den Logarithmus mit der Basis 2.)

Es sei $\lambda(x)$ eine monoton wachsende, von unten konkave Funktion mit $\lambda(1)\geqq 1$, $\lim\limits_{x\to\infty} \lambda(x)=\infty$ und

(6) $$\lambda(x)\leqq C_{12}\log^2 x \quad (C_{12}\geqq 1;\ x\geqq 2).$$

Wir werden eine Indexfolge $\{m_k\}$ definieren. Es sei $m_1=1$, und m_{k+1} sei die kleinste natürliche Zahl mit $\lambda(m_{k+1}) > 2\lambda(m_k+1)$ $(k=1, 2, \ldots)$. Wegen der Konkavität gilt

$$\frac{\lambda(2m_k)-\lambda(m_k+1)}{m_k-1}\leqq\frac{\lambda(m_k+1)-\lambda(m_1)}{m_k-m_1+1},$$

woraus

$$\lambda(2m_k)-\lambda(m_k+1)\leqq\frac{m_k-m_1}{m_k}\lambda(m_k+1)\leqq\lambda(m_k+1)$$

folgt. Nach der Definition von m_{k+1} gilt also $m_{k+1}>2m_k$ $(k\geqq 2)$. Daraus erhalten wir

$$C_{12}\log^2(m_{k+1}-m_k) > C_{12}\log^2 m_k \geqq \lambda(m_k) \qquad (k=2, 3, \ldots).$$

Ist k genügend groß $(k>k_0)$, dann gelten

$$\lambda(m_k) \leqq \lambda(m_k-1)+1, \quad \lambda(m_k)/C_{10}C_{12}\geqq 8,$$

und es gibt eine natürliche Zahl $\bar{q}_k$ mit $m_{k+1}-m_k > 2\bar{q}_k$ und

$$4C_{10}C_{12}\leqq\frac{\lambda(m_k)}{2}\leqq C_{10}C_{12}\log^2\left[\frac{m_{k+1}-m_k}{2\bar{q}_k}\right]\leqq\lambda(m_k).$$

($[\alpha]$ bezeichnet den ganzen Teil von α; die Konstanten sind in Hilfssatz II, bzw. in (6) definiert.) Es seien $n_0=1$, $n_k=m_{k+k_0}$, $q_k=\bar{q}_{k+k_0}$ $(k=1, 2, \ldots)$. Dann gelten

(7) $$n_{k+1}>2n_k \qquad (k=1, 2, \ldots),$$

(8) $$\lambda(n_{k+1}) \leqq 7\lambda(n_k-1) \qquad (k=1, 2, \ldots),$$

(9) $$4C_{10}C_{12}\leqq\frac{\lambda(n_k)}{2}\leqq C_{10}C_{12}\log^2\left[\frac{n_{k+1}-n_k}{2q_k}\right]\leqq\lambda(n_k) \qquad (k=1, 2, \ldots).$$

Beweis des Satzes I. Für eine Folge $a=\{a_n\}_1^\infty$ und für eine natürliche Zahl N setzen wir $a(1, N)=\{a_1, \ldots, a_N, 0, \ldots\}$. Offensichtlich gilt

$$\|a(1, N);\lambda\|^* \leqq \|a(1, N+1);\lambda\|^* \to \|a;\lambda\|^* \qquad (N\to\infty).$$

So ist es genügend die Ungleichung

(10) $$C_1\left\{\sum_{n=1}^{n_{k_0}-1} a_n^2\lambda(n)\right\}^{1/2}\leqq\|a(1, n_{k_0}-1);\lambda\|^* \qquad (k_0=2, 3, \ldots)$$

zu beweisen. Weiterhin gilt $\|\{\pm a_n\};\lambda\|^*=\|\{a_n\};\lambda\|^*$ offensichtlich für jede Vorzeichensverteilung von a_n; also kann man $a_n\geqq 0$ $(n=1, 2, \ldots)$ annehmen.

Es sei also $k_0(\geqq 3)$ eine natürliche Zahl. Wir setzen

$$\varphi_n(1;x)=\chi_n(x) \qquad (n=1,\dots,n_2-1).$$

Dann gelten

$$(11) \qquad \int_0^1 \left|\sum_{k=1}^{n} \varphi_k(1;x)\varphi_k(1;t)\right| dt \leqq 1 \leqq \lambda_n \qquad (0\leqq x\leqq 1;\ n=1,\dots,n_2-1),$$

$$(12) \qquad \int_0^1 \sup_{1\leqq s\leqq t<n_2} |a_s\varphi_s(1;x)+\dots+a_t\varphi_t(1;x)|\,dx \geqq a_1 \geqq C_{13}\left\{\sum_{n=1}^{n_2-1} a_n^2\lambda(n)\right\}^{1/2}.$$

Weiterhin werden wir für jedes k $(2\leqq k\leqq k_0)$ ein in $(0,1)$ orthonormiertes System $\varphi_l(k;x)$ $(l=n_k,\dots,n_{k+1}-1)$ derart definieren, daß für jedes k gilt:

$$(13) \qquad \int_0^1 \left|\sum_{l=n_k}^{n} \varphi_l(k;x)\varphi_l(k;t)\right| dt \leqq \lambda_n \qquad (0\leqq x\leqq 1;\ n=n_k,\dots,n_{k+1}-1),$$

$$(14) \qquad \int_0^1 \sup_{n_k\leqq s\leqq t<n_{k+1}} |a_s\varphi_s(k;x)+\dots+a_t\varphi_t(k;x)|\,dx \geqq C_{14}A_k,$$

mit

$$A_k=\left\{\sum_{i=1}^{c(k)}(n_k-n_{k-1})a^2_{n_k+i(n_k-n_{k-1})}\lambda(n_k)\right\}^{1/2}, \quad c(k)=\left[\frac{n_{k+1}-n_k}{n_k-n_{k-1}}\right].$$

Wir wenden zu diesem Zwecke den Hilfssatz II im Falle

$$p=\left[\frac{n_k-n_{k-1}}{2q_{k-1}}\right], \quad q=q_{k-1}$$

an. Die entsprechenden Funktionen bezeichnen wir mit $g_s(x)$ $(s=1,\dots,2pq)$. Dann gelten auf Grund des Hilfssatzes II und der Ungleichung (9)

$$(15) \qquad \int_0^1 \left|\sum_{s=1}^{\sigma} g_s(x)g_s(t)\right| dt \leqq \lambda(n_k) \qquad (0\leqq x\leqq 1;\ s=1,\dots,2pq),$$

$$(16) \qquad \begin{aligned} &\int_0^1 \sup_{1\leqq s\leqq t\leqq 2pq} |a_{n_k+(i-1)(n_k-n_{k-1})+s-1}g_s(x)+\dots+a_{n_k+(i-1)(n_k-n_{k-1})+t-1}g_t(x)|\,dx \geqq \\ &\geqq C_{15}\sqrt{n_k-n_{k-1}}\,a_{n_k+i(n_k-n_{k-1})}\sqrt{\lambda(n_{k-1})} \qquad (i=1,\dots,c(k)).\end{aligned}$$

Es seien $s_0=0$, $s_i=\sum_{j=1}^{i} a^2_{n_k+j(n_k-n_{k-1})}\Big/2\sum_{j=1}^{c(k)} a^2_{n_k+j(n_k-n_{k-1})}$ $(i=1,\dots,c(k))$, $s_{c+1}=1$ und $I_i=(s_{i-1},s_i)$ $(i=1,\dots,c+1)$. Wir setzen

$$\varphi_n(k;x)=\begin{cases}\dfrac{1}{\sqrt{m(I_i)}}\,g_{n-(n_k+(i-1)(n_k-n_{k-1}))+1}\left(\dfrac{x-s_{i-1}}{m(I_i)}\right), & x\in I_i,\\ 0, & \text{sonst}\end{cases}$$

$$(n_k+(i-1)(n_k-n_{k-1})\leqq n<n_k+(i-1)(n_k-n_{k-1})+2pq;\quad i=1,\dots,c(k)),$$

weiterhin seien die Funktionen $\varphi_n(k, x)$ $(n_k+(i-1)(n_k-n_{k-1})+2pq < n \leqq n_k+ +i(n_k-n_{k-1});\ i=1, \dots, c(k))$ der Reihe nach gleich den Funktionen

$$f_n(x) = \begin{cases} \sqrt{2}\,\chi_n(2(x-1/2)), & x \in I_{c(k)+1}, \\ 0, & \text{sonst} \end{cases} \qquad (n=1, 2, \dots)$$

Offensichtlich bilden die Funktionen $\varphi_k(k; x)$ $(l = n_k, \dots, n_{k+1}-1)$ ein orthonormiertes System in $(0, 1)$. Aus dem Hilfssatz I und aus (15), auf Grund der Definition der Funktionen ergibt sich:

$$\int_0^1 \left| \sum_{l=n_k}^{n} \varphi_l(k; x)\varphi_l(k; t) \right| dt \leqq \begin{cases} \lambda(n_k), & x \in \bigcup_{i=1}^{c(k)} I_i, \\ 1, & x \in I_{c(k)+1} \end{cases} \qquad (n_k \leqq n < n_{k+1}),$$

also ist (13) erfüllt. Weiterhin aus (8), (16) und aus der Definition der Funktionen erhalten wir durch eine einfache Rechnung:

$$\int_0^1 \sup_{n_k \leqq s \leqq t < n_{k+1}} |a_s\varphi_s(k; x)+\dots+a_t\varphi_t(k; x)|\, dx \geqq$$

$$\geqq \sum_{i=1}^{c(k)} \int_{I_i} \sup_{\substack{n_k+(i-1)(n_k-n_{k-1}) \leqq \\ \leqq s \leqq t \leqq n_k+(i-1)(n_k-n_{k-1})+2pq}} |a_s\varphi_s(k; x)+\dots+a_t\varphi_t(k; x)|\,dx \geqq$$

$$\geqq C_{15} \sum_{i=1}^{c(k)} \sqrt{n_k-n_{k-1}}\, a^2_{n_k+i(n_k-n_{k-1})} \sqrt{\lambda(n_{k-1})}\sqrt{m(I_i)} \geqq$$

$$\geqq C_{14} \left\{ \sum_{i=1}^{c(k)} (n_k-n_{k-1}) a^2_{n_k+i(n_k-n_{k-1})}) \lambda(n_k) \right\}^{1/2},$$

also ist (14) auch erfüllt.

Endlich definieren wir ein orthonormiertes System $\varphi = \{\varphi_n(x)\}_1^\infty$ in $(0, 1)$. Es seien

$$\tilde{s}_0 = 0, \quad \tilde{s}_1 = \frac{1}{4}, \quad \tilde{s}_i = \frac{1}{4} + \sum_{k=2}^{i} A_k^2 / 2 \sum_{k=2}^{k_0-1} A_k^2 \quad (i=2, \dots, k_0-1), \quad \tilde{s}_{k_0} = 1,$$

und $\tilde{I}_k = (\tilde{s}_{k-1}, \tilde{s}_k)$ $(k=1, \dots, k_0)$. Wir setzen

$$\varphi_n(x) = \begin{cases} \dfrac{1}{\sqrt{m(\tilde{I}_k)}} \varphi_n\left(k; \dfrac{x-\tilde{s}_{k-1}}{m(\tilde{I}_k)}\right), & x \in \tilde{I}_k, \\ 0, & \text{sonst} \end{cases}$$

(für $n_0 \leqq n < n_2$ im Falle $k=1$, und für $n_k \leqq n < n_{k+1}$ im Falle $k = 2, \dots, k_0-1$, und

$$\varphi_n(x) = \begin{cases} 2\chi_{n-n_{k_0}+1}(4(x-3/4)), & x \in \tilde{I}_{k_0}, \\ 0, & \text{sonst} \end{cases} \qquad (n \geqq n_{k_0}).$$

Offensichtlich bilden die Funktionen $\varphi_n(x)$ $(n=1, 2, \ldots)$ ein orthonormiertes System in $(0, 1)$. Aus Hilfssatz I, und aus (11), (13) bekommen wir durch einfache Rechnung, daß $\varphi \in \Lambda^*$ gilt. Aus (12) und (14) folgt weiterhin:

$$\|a(1, n_{k_0}-1); \lambda\|^* \geqq \int_0^1 \sup_{1 \leqq s \leqq t < n_{k_0}} |a_s \varphi_s(x) + \ldots + a_t \varphi_t(x)| \, dx \geqq \tag{17}$$

$$\geqq \int_{\tilde{I}_1} \sup_{n_0 \leqq s \leqq t < n_2} |a_s \varphi_s(x) + \ldots + a_t \varphi_t(x)| \, dx +$$

$$+ \sum_{k=2}^{k_0-1} \int_{\tilde{I}_k} \sup_{n_k \leqq s \leqq t < n_{k+1}} |a_s \varphi_s(x) + \ldots + a_t \varphi_t(x)| \, dx \geqq$$

$$\geqq C_{16} \sqrt{\sum_{n=1}^{n_2-1} a_n^2 \lambda(n)} + C_{14} \sum_{k=2}^{k_0-1} \sqrt{m(\tilde{I}_k)} \cdot A_k \geqq C_{17} \left\{ \sum_{n=1}^{n_2-1} a_n^2 \lambda(n) + \sum_{k=2}^{k_0-1} A_k^2 \right\}^{1/2}.$$

Wegen der Monotonität der Folge a und wegen (7), (8) ergibt sich

$$\sum_{n=1}^{n_2-1} a_n^2 \lambda(n) \geqq \sum_{n=n_1}^{n_2-1} a_n^2 \lambda(n) \geqq C_{18} \sum_{n=n_2}^{n_2+(n_2-n_1)} a_n^2 \lambda(n),$$

$$\sum_{n=n_k+(n_k-n_{k-1})}^{n_{k+1}-1} a_n^2 \lambda(n) \leqq C_{19} A_k^2 \qquad (k=2, 3, \ldots),$$

$$\sum_{n=n_k}^{n_k+(n_k-n_{k-1})-1} a_n^2 \lambda(n) \leqq \frac{1}{2} (n_k - n_{k-1}) a_{n_k}^2 \lambda(n_k) \leqq C_{20} A_{k-1}^2 \qquad (k=2, 3, \ldots).$$

Daraus und aus (17) bekommen wir die Abschätzung (10).

Beweis des Satzes II. Ohne Beschränkung der Allgemeinheit können wir $a_n \geqq 0$ $(n=1, 2, \ldots)$ annehmen. Es sei $a=\{a_n\}_1^\infty$ eine Folge, und bezeichnen wir mit a_{ν_k} $(\nu_k < \nu_{k+1};\ k=1, 2, \ldots)$ die von 0 verschiedenen Glieder der Folge a. Für die Folge $\bar{a}=\{a_{\nu_k}\}_1^\infty$ gilt

$$\|\bar{a}; \lambda\|^* \leqq \|a; \lambda\|^*. \tag{18}$$

Zum Beweis von (18) können wir $\|\bar{a}; \lambda\|^*=1$ voraussetzen. Es sei $\varepsilon>0$ beliebig. Dann gibt es ein System $\varphi \in \Lambda^*$ mit

$$\|\bar{a}; \lambda\|^* - \frac{\varepsilon}{2} < \int_0^1 \sup_{1 \leqq i \leqq j} |a_{\nu_i} \varphi_i(x) + \ldots + a_{\nu_j} \varphi_j(x)| \, dx. \tag{19}$$

Es sei $\eta>0$ derart gewählt, daß

$$\left(1-\sqrt{1-\eta}\right) \int_0^1 \sup_{1 \leqq i \leqq j} |a_{\nu_i} \varphi_i(x) + \ldots + a_{\nu_j} \varphi_j(x)| \, dx < \frac{\varepsilon}{2}. \tag{20}$$

Die Indizes n mit $a_n=0$ bezeichnen wir der Reihe nach mit $\mu_1<\dots<\mu_k<\dots$. Wir setzen

$$\psi_{\nu_i}(x)=\begin{cases}\frac{1}{\sqrt{1-\eta}}\varphi_i\left(\frac{x}{1-\eta}\right), & 0<x<1-\eta,\\ 0, & \text{sonst}\end{cases}\qquad (i=1,2,\dots),$$

$$\psi_{\mu_i}(x)=\begin{cases}\frac{1}{\sqrt{\eta}}\chi_i\left(\frac{x-(1-\eta)}{\eta}\right), & 1-\eta<x<1,\\ 0, & \text{sonst}\end{cases}\qquad (i=1,2,\dots).$$

Offensichtlich ist $\psi=\{\psi_n(x)\}_1^\infty$ ein orthonormiertes System in (0, 1), weiterhin gilt $\psi\in\Lambda^*$. Aus (19) und (20) folgt

$$\|\bar{a};\lambda\|^*-\varepsilon\leqq\int_0^1\sup_{1\leqq i\leqq j}|a_i\psi_i(x)+\dots+a_j\psi_j(x)|\,dx\leqq\|a;\lambda\|.^*$$

Da $\varepsilon>0$ beliebig ist, erhalten wir (18). Auf Grund von (18) können wir auch $a_n>0$ $(n=1,2,\dots)$ voraussetzen. Da offensichtlich

$$\sum_{n=1}^{n_{k_0}-1}a_n^2\lambda\left(\sum_{k=1}^{n_{k_0}-1}a_k^2/a_n^2\right)\to\sum_{n=1}^{\infty}a_n^2\lambda\left(\sum_{k=1}^{\infty}a_k^2/a_n^2\right)\quad(k_0\to\infty)$$

gilt, ist es genügend die Abschätzung

$$C_3\sum_{n=1}^{n_{k_0}-1}a_n^2\lambda\left(\sum_{k=1}^{n_{k_0}-1}a_k^2/a_n^2\right)\leqq\|a(1,n_{k_0}-1);\lambda\|^*\tag{21}$$

für ein beliebige natürliche Zahl $k_0(\geqq 3)$ zu beweisen. Ohne Beschränkung der Allgemeinheit können wir annehmen, daß

$$\sum_{n=1}^{n_{k_0}-1}a_n^2=1.\tag{22}$$

Es sei also $k_0(\geqq 3)$ eine natürliche Zahl. Es sei weiterhin $\{a_{\nu_l}\}$ $(l=1,\dots,n_{k_0}-1)$ die monoton abnehmende Anordnung der Folge a_n $(n=1,\dots,n_{k_0}-1)$ (diejenigen Glieder, die miteinander gleich sind, bleiben in der originellen Reihenfolge). Für ein k $(2\leqq k<k_0)$ und für ein i $(i=1,\dots,c(k))$ bezeichnen wir mit $Z_i(k)$ die Menge der natürlichen Zahlen ν_l mit

$$n_k+(i-1)(n_k-n_{k-1})\leqq l<n_k+i(n_k-n_{k-1}).$$

Die Anzahl der Elemente von $Z_i(k)$ ist n_k-n_{k-1}. Auf Grund von (7) gibt es $[(n_k-n_{k-1})/2]+1$ Elemente ν_l von $Z_i(k)$ derart, daß $\nu_l\geqq n_{k-1}$. Die Menge dieser Elemente bezeichnen wir mit $\bar{Z}_i(k)$; die Elemente von $\bar{Z}_i(k)$ bezeichnen wir mit

$m_l(i,k)$ $\big(m_l(i,k)<m_{l+1}(i,k);\ l=1,\dots,[(n_k-n_{k-1})/2]+1\big)$. Also gilt

(23) $$n_{k-1}\leqq m_l(i,k) \qquad \big(l=1,\dots,[(n_k-n_{k-1})/2]+1;\qquad i=1,\dots,c(k)\big).$$

Es sei Z_1 die Menge der Indizes n mit $1\leqq n$ und $n\notin \bigcup_{k=2}^{k_0-1}\bigcup_{i=1}^{c(k)}\bar Z_i(k)$; die Elemente von Z_1, bezeichnen wir mit $\mu_1<\dots<\mu_s<\cdots$. Wir setzen

$$\varphi_{\mu_i}(x)=\begin{cases}\sqrt{2}\,\chi_i(2x), & 0<x<1/2\\ 0, & \text{sonst}\end{cases} \qquad (i=1,2,\dots).$$

Diese Funktionen bilden ein orthonormiertes System in $(0,1)$, weiterhin gelten

(24) $$\int_0^1 \Big|\sum_{\substack{i\\ \mu_i\leqq n}} \varphi_{\mu_i}(x)\varphi_{\mu_i}(t)\Big|\,dt\leqq\begin{cases}1, & 0\leqq x<1/2,\\ 0, & 1/2<x<1\end{cases} \qquad (n=1,2,\dots),$$

(25) $$\int_0^1 \sup_{1\leqq i\leqq j}|a_{\mu_i}\varphi_{\mu_i}(x)+\dots+a_{\mu_j}\varphi_{\mu_j}(x)|\,dx=$$
$$=\frac{1}{\sqrt{2}}\int_0^1 \sup_{1\leqq i\leqq j}|a_{\mu_i}\chi_i(x)+\dots+a_{\mu_j}\chi_j(x)|\,dx\geqq\frac{1}{\sqrt{2}}a_{\nu_1}\geqq C_{21}\left\{\sum_{l=1}^{n_2-1}a_{\nu_l}^2\lambda(l)\right\}^{1/2}.$$

Für eine natürliche Zahl k $(2\leqq k<k_0)$ bezeichnen wir die Elemente von $\bigcup_{i=1}^{c(k)}\bar Z_i(l)$ mit $m_l(k)$ $\big(k=1,\dots,([(n_k-n_{k-1})/2]+1)c(k)\big)$, weiterhin sei

$$A_k^*=\sum_{i=1}^{c(k)}(n_k-n_{k-1})\min_{j\in Z_i(k)}a_j^2\cdot\lambda(n_k).$$

Für jede natürliche Zahl k $(2\leqq k<k_0)$ werden wir ein in $(0,1)$ orthonormiertes System $\bar\varphi_{m_l(k)}(k;x)$ $\big(l=1,\dots,([(n_k-n_{k-1})/2]+1)c(k)\big)$ derart definieren, daß

(26) $$\int_0^1\Big|\sum_{l=1}^{\lambda}\bar\varphi_{m_l(k)}(k;x)\bar\varphi_{m_l(k)}(k;t)\Big|\,dt\leqq\lambda(m_{k-1}) \qquad (0<x<1)$$
$$(\lambda=1,\dots,([(n_k-n_{k-1})/2]+1)c(k)),$$

(27)
$$\int_0^1 \sup_{1\leqq s\leqq t\leqq([(n_k-n_{k-1})/2]+1)c(k)}|a_{m_s(k)}\bar\varphi_{m_s(k)}(k;x)+\dots+a_{m_t(k)}\bar\varphi_{m_t(k)}(k;x)|\,dx\geqq C_{22}A_k^*.$$

Es sei k eine natürliche Zahl $(2\leqq k<k_0)$. Das System

$$\bar\varphi_{m_l(k)}(k;x) \qquad \big(l=1,\dots,([(n_k-n_{k-1})/2]+1)c(k)\big)$$

definieren wir folgenderweise. Wir wenden den Hilfssatz II im Falle

$$\bar p=\left[\left(\left[\frac{n_k-n_{k-1}}{2}\right]+1\right)\Big/2q_{k-1}\right],\quad \bar q=q_{k-1}$$

an. Die entsprechenden Funktionen bezeichnen wir mit $\bar{g}_s(x)$ $(s=1, \ldots, 2\bar{p}\bar{q})$. Dann gelten auf Grund des Hilfssatzes II und der Ungleichung (9)

$$\int_0^1 \left| \sum_{s=1}^{\sigma} \bar{g}_s(x)\bar{g}_s(t) \right| dt \leqq \lambda(n_{k-1}) \quad (0<x<1; \sigma=1, \ldots, 2\bar{p}\bar{q}), \tag{28}$$

$$\int_0^1 \sup_{1\leqq s\leqq t\leqq 2\bar{p}\bar{q}} |a_{m_s(i,k)}\bar{g}_s(x)+\ldots+a_{m_t(i,k)}\bar{g}_t(x)|\, dx \geqq \tag{29}$$

$$\geqq C_{23}\sqrt{n_k-n_{k-1}} \min_{j\in Z_i(k)} a_j \cdot \sqrt{\lambda(n_k)} \quad (i=1, \ldots, c(k)).$$

Es seien, $\bar{s}_0=0$, $\bar{s}_i = \sum_{j=1}^{i} \min_{t\in Z_j(k)} a_t^2/2 \sum_{j=1}^{c(k)} \min_{t\in Z_j(k)} a_t^2$ $(i=1, \ldots, c(k))$, $\bar{s}_{c(k)+1}=1$, und $\bar{I}_i=(\bar{s}_{i-1}, \bar{s}_i)$ $(i=1, \ldots, c(k))$. Wir setzen

$$\bar{\varphi}_{m_l(i,k)}(k; x) = \begin{cases} \dfrac{1}{\sqrt{m(\bar{I}_i)}} g_l\left(\dfrac{x-\bar{s}_{i-1}}{m(\bar{I}_i)}\right), & x\in \bar{I}_i, \\ 0, & \text{sonst} \end{cases}$$

$(l=1, \ldots, 2\bar{p}\bar{q};\ i=1, \ldots, c(k))$, weiterhin seien die Funktionen

$$\bar{\varphi}_{m_l(i,k)}(k; x) \qquad (l=2\bar{p}\bar{q}+1, \ldots, [(n_k-n_{k-1})/2]+1; \qquad i=1, \ldots, c(k))$$

der Reihe nach gleich den Funktionen

$$\bar{f}_n(x) = \begin{cases} \sqrt{2}\,\chi_n(2(x-1/2)), & 1/2<x<1, \\ 0, & \text{sonst} \end{cases} \qquad (n=1, 2, \ldots).$$

Offensichtlich bilden die Funktionen $\varphi_{m_l(k)}(k; x)$ in $(0, 1)$ ein orthonormiertes System. Aus dem Hilfssatz I und aus (28), auf Grund der Definition der Funktionen ergibt sich für $\lambda=1, \ldots, ([(n_k-n_{k-1})/2]+1)c(k)$

$$\int_0^1 \left| \sum_{l=1}^{\lambda} \bar{\varphi}_{m_l(k)}(k; x)\bar{\varphi}_{\bar{m}_l(k)}(k; t) \right| dt \leqq \lambda(n_{k-1}) \quad (0\leqq x\leqq 1),$$

also ist (26) erfüllt. Weiterhin aus (8), (29) und aus der Definition dieser Funktionen erhalten wir durch eine einfache Rechnung

$$\int_0^1 \sup_{1\leqq s\leqq t\leqq ([(n_k-n_{k-1})/2]+1)c(k)} |a_{m_s(k)}\varphi_{m_s(k)}(k; x)+\ldots+a_{m_t(k)}\bar{\varphi}_{m_t(k)}(k; x)|\, dx \geqq$$

$$\geqq \sum_{i=1}^{c(k)} \int_{\bar{I}_i} \sup_{1\leqq s\leqq t\leqq 2\bar{p}\bar{q}} |a_{m_s(i,k)}\bar{\varphi}_{m_s(i,k)}(k; x)+\ldots+a_{m_t(i,k)}\bar{\varphi}_{m_t(i,k)}(k; x)|\, dx \geqq$$

$$\geqq C_{23} \sum_{i=1}^{c(k)} \sqrt{m(\bar{I}_i)}\sqrt{n_k-n_{k-1}} \min_{j\in Z_i(k)} a_j \sqrt{\lambda(n_k)} \geqq C_{22} A_k^*,$$

also ist auch (27) erfüllt.

Es sei $s_0^* = \frac{1}{2}$, $s_i^* = \frac{1}{2} + \sum_{j=1}^{i} A_j^{*2}/2 \sum_{j=1}^{k_0-1} A_j^{*2}$, $I_i^* = (s_{i-1}^*, s_i^*)$ $(i=1, \dots, k_0-1)$. Wir setzen für $l = 1, \dots, \big([(n_k - n_{k-1})/2]+1\big)c(k)$; $k=2, \dots k_0-1$,

$$\varphi_{m_l(k)}(x) = \begin{cases} \frac{1}{\sqrt{m(I_k^*)}} \bar{\varphi}_{m_l(k)}\left(\frac{x - s_{k-1}^*}{m(I_k^*)}\right), & x \in I_k^*, \\ 0 \text{ sonst} \end{cases}$$

Offensichtlich bilden die Funktionen $\varphi_n(x)$ $(n = 1, \dots, n_{k_0}-1)$ ein orthonormiertes System in $(0, 1)$. Aus (25), (27) ergibt sich

$$(30) \qquad \int_0^1 \sup_{1 \le i \le j < n_{k_0}} |a_i \varphi_i(x) + \dots + a_j \varphi_j(x)| \, dx \ge C_{24} \left\{ \sum_{n=1}^{n_2-1} a_{\nu_n}^2 \lambda(n) + \sum_{k=2}^{k_0-1} A_k^{*2} \right\}^{1/2}.$$

Weiterhin aus (24) und (26) bekommen wir

$$(31) \qquad \int_0^1 \left| \sum_{l=1}^{n} \varphi_l(x) \varphi_l(t) \right| dt \le \begin{cases} 1 \le \lambda_n, & x \in (0, 1) \setminus \bigcup_{i=1}^{k_0-1} I_i^*, \quad n = 1, 2, \dots, \\ \lambda(n_{k-1}), & x \in I_k^*, \quad n \ge \min_{i,l} m_l(i, k), \quad k = 2, \dots, k_0-1, \\ 0, & x \in I_k^*, \quad n < \min_{i,l} m_l(i, k), \quad k = 2, \dots, k_0-1. \end{cases}$$

Wegen (8) und (23) gilt

$$(32) \qquad \lambda(n_{k-1}) \le \lambda(n) \quad (n \ge \min_{i,l} m_l(i, k); \quad k = 2, \dots, k_0-1).$$

Aus (32) erhalten wir $\varphi \in A$.

So bekommen wir aus (30)

$$(33) \quad \|a(1, n_{k_0}-1); \lambda\|^* \ge C_{24} \left\{ \sum_{n=1}^{n_2-1} a_{\nu_n}^2 \lambda(n) + \sum_{k=2}^{k_0-1} \sum_{i=1}^{c(k)} (n_k - n_{k-1}) \min_{j \in Z_i(k)} a_j^2 \cdot \lambda(n_k) \right\}^{1/2}.$$

Wegen der Monotonität der Folge a und wegen (7), (8) ergibt sich

$$\sum_{n=1}^{n_2-1} a_{\nu_n}^2 \lambda(n) \ge \sum_{n=n_1}^{n_2-1} a_{\nu_n}^2 \lambda(n) \ge C_{25} \sum_{n=n_2}^{n_2+(n_2-n_1)-1} a_{\nu_n}^2 \lambda(n),$$

$$\sum_{n=n_k+(n_k-n_{k-1})}^{n_{k+1}-1} a_{\nu_n}^2 \lambda(n) \le C_{26} A_k^{*2} \qquad (k = 2, \dots, k_0-1),$$

$$\sum_{n=n_k}^{n_k+(n_k-n_{k-1})-1} a_{\nu_n}^2 \lambda(n) \le C_{27} (n_k - n_{k-1}) \min_{j \in Z_{c(k)}(k-1)} a_j^2 \cdot \lambda(n_k) \le C_{28} A_{k-1}^{*2}$$
$$(k = 2, \dots, k_0-1).$$

Daraus und aus (33) bekommen wir die Abschätzung

$$(34) \qquad \|a(1, n_{k_0}-1); \lambda\|^* \ge C_{29} \left\{ \sum_{n=1}^{n_{k_0}-1} a_{\nu_n}^2 \lambda(n) \right\}^{1/2}$$

Wegen $\sum_{n=1}^{n_{k_0}-1} a_{\nu_n}^2=1$ und wegen $a_{\nu_n} \geqq a_{\nu_{n+1}}$ $(n = 1, \dots, n_{k_0}-2)$ gilt $1/a_{\gamma_n}^2 \geqq n$. Es seien $l_1<\dots<l_\lambda(<n_{k_0})$ diejenigen Indizes, für die $1/a_{\nu_{l_n}}^2 \leqq l_n^2$ gilt; die anderen Indizes zwischen 1 und $n_{k_0}-1$ bezeichnen wir mit $p_1<\dots<p_r$; für diese Indizes gilt also $1/a_{\nu_{p_n}}^2 > p_n^2$. Auf Grund der Voraussetzungen über die Funktion $\lambda(x)$ folgt durch eine einfache Rechnung

$$\sum_{n=1}^{n_{k_0}-1} a_{\nu_n}^2 \lambda(1/a_{\nu_n}^2) = \sum_{s=1}^{\lambda} a_{\nu_{l_s}}^2 \lambda(1/a_{\nu_{l_s}}^2) + \sum_{t=1}^{r} a_{\nu_{p_t}}^2 \lambda(1/a_{\nu_{p_t}}^2) \leqq$$

$$\leqq \sum_{s=1}^{\lambda} a_{\nu_{l_s}}^2 \lambda(l_s^2) + C_{12} \sum_{t=1}^{\tau} \frac{1}{p_t^{3/2}} a_{\nu_{p_t}}^{1/2} \log^2(1/a_{\nu_{p_t}}^2) \leqq C_2 \sum_{s=1}^{\lambda} a_{\nu_{l_s}}^2 \lambda(l_s) + C_{30} \sum_{t=1}^{n_{k_0}-1} \frac{1}{t^{3/2}} \leqq$$

$$\leqq C_{31} \left(\sum_{n=1}^{n_{k_0}-1} a_{\nu_n}^2 \lambda(n) + \sum_{n=1}^{n_{k_0}-1} a_{\nu_n}^2 \right) \leqq C_{32} \sum_{n=1}^{n_{k_0}-1} a_{\nu_n}^2 \lambda(n).$$

Zusammen mit (34) führt dies zur Abschätzung (21).

Damit haben wir auch Satz II bewiesen.

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Operators satisfying a sequential growth condition

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§ 1. Introduction

An operator T on a Hilbert space $\mathfrak{H}$ is called hyponormal if $T^*T-TT^* \geqq 0$. One of the very useful properties of a hyponormal operator T is that it satisfies the G_1 *growth condition*, that is $\|(\lambda-T)^{-1}\| = 1/d(\lambda)$ for all $\lambda \in \varrho(T)$ where $\sigma(T)$ and $\varrho(T)$ denote the spectrum and the resolvent set of T respectively, and $d(\lambda)= =\text{dist}\,[\lambda, \sigma(T)]$. For most applications we need this growth condition to be satisfied in a neighborhood of $\sigma(T)$. On the other hand, the Volterra operator V does not satisfy the growth condition G_1 in any neighborhood of $\sigma(V)$, but there does exist a sequence $\lambda_n \in \varrho(V)$ (take λ_n to be negative real numbers) such that $\lambda_n \to 0$ and $\|(V-\lambda_n)^{-1}\| = 1/|\lambda_n|$. This motivates us to introduce the concept of a *sequential G_1 growth condition*. A bounded operator T on a Hilbert space $\mathfrak{H}$ satisfies sequential G_1 growth condition if for every $\lambda \in \partial(\sigma(T))$ (the boundary of $\sigma(T)$), there exists a sequence $\lambda_n \in \varrho(T)$ such that $\lambda_n \to \lambda$ and $\|(\lambda_n-T)^{-1}\| = 1/d(\lambda_n)$ for all n. Such an operator T is also referred to as a *sequentially G_1 operator*. Some other generalizations of G_1 growth conditions have been considered by LUCKE [5, 6] and RIGGS [8].

We prove that a sequentially G_1 algebraic operator is normal. This result has an interesting application to the theory of ϱ-dilations in the sense that it generalizes and at the same time simplifies the proof of a recent theorem of FURUTA [2] concerning C_ϱ-operators. We also prove that if T is a sequentially G_1 operator then $T+\mathcal{K} \subset \overline{\mathcal{R}}_1$ where $\mathcal{K}$ is the ideal of compact operators and $\overline{\mathcal{R}}_1$ denotes the norm closure of operators with one dimensional reducing subspace. Our result generalizes a theorem of BERBERIAN [1] and ISTRĂŢESCU [4] which asserts that $T+\mathcal{K} \subset \overline{\mathcal{R}}_1$ whenever T is a G_1 operator (this in turn is a generalization of a result of STAMPFLI [12] about hyponormal operators). $\mathcal{B}(\mathfrak{H})$ denotes the algebra of bounded linear operator on $\mathfrak{H}$.

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The following proposition shows the existence of a class of sequentially G_1 operators which are not G_1 operators.

Proposition 1.1. *If $T\in\mathcal{B}(\mathfrak{H})$ is a quasi-nilpotent operator such that $0\in\partial w(T)$, where $w(T)=\{(Tx,x):x\in H$ and $\|x\|=1\}$ is the numerical range of T, then T is a sequentially G_1 operator.*

Proof. Since $w(T)$ is convex there is a line of support for $w(T)$ passing through 0 (since $0\in\partial w(T)$) and hence without loss of generality we can assume that $w(T)\subset \subset\{\lambda:\text{Real }\lambda\geqq 0\}$. It is quite easy to show that, for any $T\in\mathcal{B}(\mathfrak{H})$ and $\lambda\notin\overline{w(T)}$, $\|(\lambda-T)^{-1}\|\leqq\dfrac{1}{\operatorname{dist}[\lambda,\overline{w(T)}]}$. Since $0\in\partial w(T)$ and $\sigma(T)=\{0\}$, for any real negative number λ, $\operatorname{dist}[\lambda,\overline{w(T)}]=|\lambda|=d(\lambda)$. Hence we can take $\lambda_n=-1/n$ and then $\|(\lambda_n-T)^{-1}\|=\dfrac{1}{|\lambda_n|}$ for all n.

In view of a theorem of Stampfli [9], if T is a G_1 operator and if $\sigma(T)$ is a finite set, then T is a normal operator. Thus no non-zero quasi-nilpotent operator is a G_1 operator. Out next result shows that no non-zero nilpotent operator is a sequentially G_1 operator.

Proposition 1.2. *Let $T\in\mathcal{B}(\mathfrak{H})$ be such that $T^m=0$ for some $m>1$ and suppose that T is a sequentially G_1 operator. Then $T=0$.*

Proof. Since T is a sequentially G_1 operator and $\sigma(T)=\{0\}$, there exists a sequence $\lambda_n\to 0$ such that

$\|(\lambda_n-T)^{-1}\|=\dfrac{1}{|\lambda_n|}$ for all n. Suppose $m>1$, then $(\lambda_n-T)^{-1}=\sum_{i=0}^{m-1}\dfrac{T^i}{\lambda_n^{i+1}}$ this implies $\dfrac{\|T^{m-1}\|}{|\lambda_n|^m}-\sum_{i=0}^{m-2}\dfrac{\|T^i\|}{|\lambda_n|^{i+1}}\leqq\dfrac{1}{|\lambda_n|}$ for all n. Hence $\|T^{m-1}\|\leqq|\lambda_n|^{m-1}+\sum_{i=0}^{m-2}\|T^i\|\,|\lambda_n|^{m-i-1}$, for all n. Since $|\lambda_n|^{m-1}+\sum_{i=0}^{m-2}\|T^i\|\,|\lambda_n|^{m-i-1}\to 0$ as $n\to\infty$, we conclude that $T^{m-1}=0$. Hence by a simple induction argument $T=0$. We thank the Referee for pointing out that this result holds even if $\|(\lambda_n-T)^{-1}\|\leqq\dfrac{M}{|\lambda_n|}$, $M\geqq 1$.

Corollary 1.3. *Let $T\neq 0$ be a nilpotent operator, then $0\in$ Interior $w(T)$.*

§ 2. Sequentially G_1 operators and the class C_ϱ

An operator T is called *algebraic* if there exists a polynomial $p(z)$ such that $p(T)=0$. We assume that this $p(z)$ is minimal among all the polynomials $q(z)$ such that $q(T)=0$. We shall show that if T is a sequentially G_1 algebraic operator then T

is normal. To prove this result we need the following lemma, which appears implicitly in STAMPFLI [10] and explicitly in PUTNAM [7] and STAMPFLI [12].

Lemma 2. 1. (Putnam—Stampfli) *Let* $T\in\mathscr{B}(\mathfrak{H})$ *and let* $\lambda_0\in\sigma(T)$ *such that* $Tx=\lambda_0 x$, $\|x\|=1$. *Suppose there exists a sequence* $\{\lambda_n\}\in\varrho(T)$ *such that* $\lambda_n\to\lambda_0$ *and* $\lim_{n\to\infty}|\lambda_n-\lambda_0|\,\|(T-\lambda_n)^{-1}\|=1$; *then* $T^*x=\bar{\lambda}_0 x$.

Theorem 2. 2. *Let* $T\in\mathscr{B}(\mathfrak{H})$ *be a sequentially* G_1 *algebraic operator. Then* T *is normal.*

Proof. Since T is algebraic, there exists a polynomial $p(z)$ such that $p(T)=0$. Let z_i $(i=0,\ldots,m)$ be the distinct roots of $p(z)$ of multiplicity n_i $(i=0,\ldots,m)$ respectively. Then $\mathfrak{H}=\sum_{i=1}^{m}\eta_i$ where $\eta_i=\{x\in\mathfrak{H}:(T-z_i)^{n_i}x=0\}$. Thus each η_i is invariant under T and $\sigma(T|\eta_i)=\{z_i\}$. Since T is sequentially G_1, it follows that $T|\eta_i$ is sequentially G_1. Moreover $T-z_i|\eta_i$ is a nilpotent operator. Hence by Proposition 1. 2, $T-z_i|\eta_i=0$. Thus $\eta_i=\eta(T-z_i)=$ null space of $T-z_i$. Moreover, by Lemma 2. 1, $\eta(T-z_i)=\eta(T^*-\bar{z}_i)$ and $\eta(T-z_i)\perp\eta(T-z_j)$ for $i\neq j$. Hence $T=\sum_{i=0}^{m}\oplus z_iP_i$ where P_i denotes the projection of $\mathfrak{H}$ onto $\eta(T-z_i)$ and T is normal.

The next theorem shows that the above hypothesis can be slightly changed without affecting the conclusion. The hypothesis in the following theorem means roughly that T is sequentially G_1 except at one point.

Theorem 2. 3. *Let* $T\in\mathscr{B}(\mathfrak{H})$ *such that* $p(T)=0$, *where* $p(z)=(z-z_0)(z-z_1)^{n_1}\ldots$ $\ldots(z-z_m)^{n_m}$. *Suppose for each* z_i $(i=1,2,\ldots,m)$ *there exists a sequence* $\{\lambda_n^{(i)}\}_{n=0}^{\infty}\in\varrho(T)$ *such that* $\lambda_n^{(i)}\to z_i$ *and* $\|(\lambda^{(i)}-T)^{-1}\|=\dfrac{1}{|\lambda_n^{(i)}-z_i|}$ *for all* n. *Then* $\mathfrak{H}=\sum_{i=0}^{m}\oplus\eta(T-z_i)$ *and* T *is normal.*

Proof. From the proof of Theorem 2. 2, it follows that

$$\mathfrak{H}=\eta(T-z_0)\dotplus\sum_{i=1}^{m}\oplus\eta(T-z_i)\quad\text{and}\quad\eta(T-z_i)=\eta(T^*-\bar{z}_i)\quad\text{for}\quad i=1,2,\ldots,m.$$

Thus $\eta(T-z_0)$ is also orthogonal to $\eta(T-z_i)$ for $i=1,2,\ldots,m$. Hence T is normal.

Now we shall apply the above result to get a generalization of a result of FURUTA [2] about the operators in C_ϱ class. The *class* C_ϱ of operators was introduced by SZ.-NAGY and FOIAŞ [13] as the set of all operators T on a Hilbert space $\mathfrak{H}$ for which there exists a unitary operator U on some Hilbert space $\mathscr{K}$ $(\mathscr{K}\supset\mathfrak{H})$ such that

$$T^n=\varrho PU^n|\mathfrak{H}\qquad(n=1,2,\ldots),$$

where P is the projection of $\mathscr{K}$ onto $\mathfrak{H}$. U is called *unitary* ϱ*-dilation* of T.

One of the characterizations of the class C_ϱ, $\varrho \geqq 2$ is the following:

Theorem 2. 4. (Sz.-Nagy and Foiaş [14]) *An operator $T \in \mathscr{B}(\mathfrak{H})$ belongs to the class C_ϱ ($\varrho \geqq 2$) if and only if T satisfies the following condition*

$$\|(\mu - T)^{-1}\| \leqq \frac{1}{|\mu| - 1} \quad \begin{array}{ll} \text{for} \quad 1 < |\mu| < \infty & \text{if} \quad \varrho = 2 \\ \text{for} \quad 1 < |\mu| \leqq \dfrac{\varrho - 1}{\varrho - 2} & \text{if} \quad \varrho > 2. \end{array}$$

Theorem 2. 5. *Let $T \in C_\varrho$ ($\varrho > 0$). Suppose $p(T) = 0$ where $p(z)$ is a polynomial and all roots of $p(z)$ are on the unit circle except for, perhaps a simple root (say z_0). Then $T = U \oplus z_0 P$ where P is a projection of $\mathfrak{H}$ onto the null space of $T - z_0$, and U is a unitary operator.*

Proof. Since $C_\varrho \subset C_{\varrho'}$ for $0 < \varrho < \varrho'$ ([14, page 50]), $T \in C_\varrho$ ($\varrho > 0$) implies that $T \in C_{\varrho+2}$ and hence by Theorem 2. 4,

$$\|(\mu - T)^{-1}\| \leqq \frac{1}{|\mu| - 1} \quad \text{for} \quad 1 < |\mu| < \frac{\varrho + 1}{\varrho}.$$

Let $p(z) = (z - z_0)(z - z_1)^{n_1} \dots (z - z_m)^{n_m}$ where $|z_i| = 1$ for $i = 1, 2, \dots, m$. Now for any μ, $1 < |\mu| < 1 + \dfrac{1}{\varrho}$, μ collinear with z_i ($i = 1, 2, \dots, m$);

$$\|(\mu - T)^{-1}\| \leqq \frac{1}{|\mu| - |z_i|} = \frac{1}{|\mu - z_i|}$$

Hence $\|(\mu - T)^{-1}\| = \dfrac{1}{|\mu - z_i|} = \dfrac{1}{d(\mu)}$. Thus T satisfies the hypothesis of Theorem 2. 3, and hence $T = \sum_{i=0}^{m} \oplus z_i P_i$ where P_i is the projection of $\mathfrak{H}$ onto the null space of $T - z_i$, $i = 0, 1, \dots, m$. Since $|z_i| = 1$ for $i = 1, 2, \dots, m$; $\sum_{i=1}^{m} \oplus z_i P_i$ is a unitary operator, and thus $T = U \oplus z_0 P_0$.

Corollary 2. 6. (Furuta [2]) *If $T^k = T$ for some positive integer $k \geqq 2$ and $T \in C_\varrho$ ($\varrho > 0$) then T is the direct sum of a zero operator and a unitary operator.*

Proof. Obvious from Theorem 2. 5.

§ 3. The class $\overline{\mathscr{R}}_1$

The class $\mathscr{R}_1$ of operators was introduced by Halmos [3]. An operator T is in $\mathscr{R}_1$ if and only if T has one dimensional reducing subspace. $\overline{\mathscr{R}}_1$ denotes the norm closure of $\mathscr{R}_1$. Halmos [3] showed that every normal operator and every isometry is in $\overline{\mathscr{R}}_1$. STAMPFLI [11] showed that $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$ whenever T is either hyponormal or a Toeplitz operator, where $\mathscr{K}$ denotes the ideal of compact operators. He also showed that if the spectral radius of T is equal to the norm of T then T is in $\overline{\mathscr{R}}_1$. We shall prove that $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$ whenever T is a sequentially G_1 operator.

The following four results are well known and are stated here for easy reference.

Lemma 3. 1. *$T\in\overline{\mathscr{R}}_1$ if and only if there exists a sequence of unit vectors $x_n\in\mathfrak{H}$ such that $\|(T-\lambda)x_n\| \to 0$ and $\|(T^*-\bar\lambda)x_n\| \to 0$ for some $\lambda\in\sigma(T)$.*

Lemma 3. 2. *Let $\lambda_0\in\sigma(T)$ where $|\lambda_0|=\|T\|$. Then there exists a sequence of unit vectors $x_n\in\mathfrak{H}$ such that $\|(T-\lambda_0)x_n\| \to 0$ and $\|(T^*-\bar\lambda_0)x_n\| \to 0$. Thus $T\in\overline{\mathscr{R}}_1$.*

Lemma 3. 3. *If there exists a sequence of unit vectors $x_n\in\mathfrak{H}$ such that x_n converges weakly to 0 and $\|(T-\lambda)x_n\| \to 0$ and $\|(T^*-\bar\lambda)x_n\| \to 0$, then $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$.*

Lemma 3. 4. *If there exists a sequence of unit vectors $x_n\in\mathfrak{H}$ such that x_n converges weakly to x_0 and $\|Tx_n\| \to 0$ then $Tx_0=0$.*

For any operator T on a Hilbert space $\mathfrak{H}$, let $\gamma_T=\{\lambda\in\sigma(T)$: there exists $x\in\mathfrak{H}$, $x\neq 0$ such that $(T-\lambda)x = 0$ and $(T^*-\bar\lambda)x = 0\}$. If γ_T is not empty then $T\in\overline{\mathscr{R}}_1$. Also if γ_T is an infinite set then it can be easily shown that $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$. In order to prove our result, we need the following lemmas.

Lemma 3. 5. *Suppose there exists a sequence of unit vectors $x_n\in\mathfrak{H}$ and $\lambda_0\notin\gamma_T$ such that $\|(T-\lambda_0)x_n\| \to 0$ and $\|(T^*-\bar\lambda_0)x_n\| \to 0$ then $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$.*

Proof. Since $\{x_n\}$ is a bounded sequence, we assume, without loss of generality, that x_n converges weakly to x_0. Then by Lemma 3. 4, $(T-\lambda_0)x_0 = 0$ and $(T^*-\bar\lambda_0)x_0 = 0$. Since $\lambda_0\notin\gamma_T$ therefore $x_0=0$. Thus by Lemma 3. 3 $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$.

Lemma 3. 6. *Let T be sequentially G_1 and suppose that γ_T is a finite set such that $\gamma_T=\sigma(T)$. Then $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$.*

Proof. Let $\sigma(T)=\{\lambda_1,\lambda_2,\dots,\lambda_n\}$. Since $\gamma_T=\sigma(T)$, each λ_i is an eigenvalue of T. Also T is sequentially G_1, therefore by Lemma 2. 1, each λ_i is a reducing eigenvalue. If for some i, $\eta(T-\lambda_i)$ which is equal to $\eta(T^*-\bar\lambda_i)$ is infinite dimensional, then obviously $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$. Otherwise we have $\mathfrak{H} = \mathfrak{H}_1\oplus\mathfrak{H}_2$ where $\mathfrak{H}_1 = \sum_{i=1}^{n}\oplus\eta(T-\lambda_i)$ and $\mathfrak{H}_2 = \mathfrak{H}\ominus\mathfrak{H}_1$ is infinite dimensional. Since $\mathfrak{H}_1$ reduces T and $\sigma(T)$ is a finite set it is not hard to verify that $T|\mathfrak{H}_2=T_2$ is a sequentially G_1 operator and γ_{T_2} is empty. Note that $T_2+\mathscr{K} \subset \overline{\mathscr{R}}_1$ implies $T+\mathscr{K} \subset \overline{\mathscr{R}}_1$ and thus this case will be considered in the proof of the next theorem.

Theorem 3.7. *If T is a sequentially G_1 operator, then $T+\mathcal{K} \subset \overline{\mathcal{R}}_1$.*

Proof. In view of Lemma 3.6 we only need to consider the case when γ_T is a finite set and $\gamma_T \neq \sigma(T)$.

Since T is sequentially G_1, for any $\lambda_0 \in \partial(\sigma(T) \setminus \gamma_T)$, there exists a sequence $\mu_n \in \varrho(T)$ such that $\|(T-\mu_n)^{-1}\| = \frac{1}{d(\mu_n)}$, and $\mu_n \to \lambda_0$. Since $\mu_n \to \lambda_0 \notin \gamma_T$, therefore for any μ_{m_0} such that $|\mu_{m_0} - \lambda_0| < \min\{|\lambda_0 - \alpha| : \alpha \in \gamma_T\}$, $d(\mu_{m_0}) = |\mu_{m_0} - \lambda_{m_0}|$ where $\lambda_{m_0} \in \sigma(T) \setminus \gamma_T$.

Thus $\|(T-\mu_{m_0})^{-1}\| = \frac{1}{d(\mu_{m_0})} = \frac{1}{|\lambda_{m_0} - \mu_{m_0}|}$, $\frac{1}{\lambda_{m_0} - \mu_{m_0}} \in \sigma((T-\mu_{m_0})^{-1})$.

Hence by Lemma 3.2, there exists a sequence of unit vectors x_n such that

$$\|[(T-\mu_{m_0})^{-1} - (\lambda_{m_0} - \mu_{m_0})^{-1}]x_n\| \to 0 \text{ and } \|[(T^* - \bar{\mu}_{m_0})^{-1} - (\bar{\lambda}_{m_0} - \bar{\mu}_{m_0})^{-1}]x_n\| \to 0.$$

Hence by the first resolvent equation we get $\|(T-\lambda_{m_0})x_n\| \to 0$ and $\|(T^* - \bar{\lambda}_{m_0})x_n\| \to 0$. Also $\lambda_{m_0} \notin \gamma_T$. Thus by Lemma 3.5, $T+\mathcal{K} \subset \overline{\mathcal{R}}_1$.

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Weighted shifts of class $\mathscr{C}_\varrho$

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§ 1. Introduction

In this paper we study weighted shifts of class $\mathscr{C}_\varrho$ and apply the results to obtain some "metric properties" of operators of class $\mathscr{C}_\varrho$. We shall include some known facts for these classes and resume parts of the papers [2] and [3].

We shall consider complex Hilbert spaces only. Operators will be supposed linear and bounded. For the Hilbert space $\mathfrak{H}$ we denote by $\mathscr{L}(\mathfrak{H})$ the algebra of all operators on $\mathfrak{H}$.

Definition. The operator $T\in\mathscr{L}(\mathfrak{H})$ is said to be of class $\mathscr{C}_\varrho$ $(\varrho>0)$ if there exist a Hilbert space $\mathfrak{K}\supset\mathfrak{H}$ and a unitary operator $U\in\mathscr{L}(\mathfrak{K})$ such that

(1) $$T^n=\varrho P_{\mathfrak{H}}U^n|_{\mathfrak{H}} \qquad (n=1, 2, \dots),$$

$P_{\mathfrak{H}}=P$ denoting orthogonal projection from $\mathfrak{K}$ onto $\mathfrak{H}$. The operator U is called the unitary ϱ-dilation of T.

The classes $\mathscr{C}_\varrho$ were introduced by B. Sz.-Nagy and C. Foiaş cf. [1]. Recall the following facts:

a) $\mathscr{C}_\varrho$ is an increasing function of ϱ, i.e. $\mathscr{C}_\varrho\supset\mathscr{C}_\sigma$ for $\varrho>\sigma$.
b) $\mathscr{C}_1$ is the class of contractions (B. Sz.-Nagy).
c) $\mathscr{C}_2$ is the class of numerical radius contractions (C. A. Berger).
d) If $T\in\mathscr{C}_\varrho$, then $\|T^n\|\leqq\varrho$ and $\nu(T)\leqq\min\{1,\varrho\}$ ($\nu(T)$ means spectral radius).
e) $T\in\mathscr{C}_\varrho$ if and only if

(2) $$(\varrho-2)\|zTh\|^2-2(\varrho-1)\operatorname{Re}(zTh, h)+\varrho\|h\|^2\geqq 0 \quad \text{for} \quad h\in\mathfrak{H} \quad \text{and} \quad |z|\leqq 1.$$

We will also use the following obvious corollaries of e):
f) If $T\in\mathscr{C}_\varrho$ and $\mathfrak{H}_0\subset\mathfrak{H}$ is a closed invariant subspace for T, then $T|\mathfrak{H}_0\in\mathscr{C}_\varrho$.
g) The class $\mathscr{C}_\varrho$ is closed in the strong operator topology.

1.1. Proposition. *If $T_j\in\mathscr{L}(\mathfrak{H}_j)$ belongs to the class $\mathscr{C}_{\varrho_j}$ $(j=1, 2)$, then $T_1\otimes T_2\in\mathscr{C}_{\varrho_1\varrho_2}$.*

Proof. Indeed, if U_j is a unitary ϱ_j-dilation of T_j in $\mathfrak{K}_j \supset \mathfrak{H}_j$, it is easy to verify that $U_1 \otimes U_2$ is a unitary $\varrho_1 \varrho_2$-dilation of $T_1 \otimes T_2$ in $\mathfrak{K}_1 \otimes \mathfrak{K}_2$.

1.2. Proposition. *$T \in \mathscr{C}_\varrho$ if and only if*

(i) $\nu(T) \leqq a = \min\{1, \varrho\}$,

(ii) $(\varrho-2)\|Th\|^2 - 2|\varrho-1|\,|(Th, h)| + \varrho\|h\|^2 \geqq 0$ *for all* $h \in \mathfrak{H}$.

(i) *is redundant if* $0 < \varrho \leqq 2$.

Proof. The case $\varrho = 1$ is obvious. The necessity part follows from d) and e) taking $|z|=1$. Let $\varrho \neq 1$ and suppose that (i) and (ii) are satisfied. Remark that (ii) may be written in the form:

$$(\varrho-2)\|zTh\|^2 - 2(\varrho-1)\,\mathrm{Re}\,(zTh, h) + \varrho\|h\|^2 \geqq 0 \quad \text{for} \quad |z|=1 \quad \text{and} \quad h \in \mathfrak{H}$$

or, equivalently,

$$\|[\varrho I - (\varrho-1)zT]h\| \geqq \|zTh\| \quad \text{for} \quad h \in \mathfrak{H}, \quad |z|=1. \tag{3}$$

From (i) it follows that $\varrho|\varrho-1|^{-1} > a$; hence

$$C(z) = zT[\varrho I - (\varrho-1)zT]^{-1} \in \mathscr{L}(\mathfrak{H}),$$

for $|z| < b$, where $b = \varrho|\varrho-1|^{-1}a^{-1}$. Since $b > 1$, inequality (3) may be written in the form

$$\|C(z)\| \leqq 1 \quad \text{for} \quad |z|=1.$$

$C(z)$ being analytic on the closed unit disk, it follows by the maximum modulus theorem that

$$\|C(z)\| \leqq 1 \quad \text{for} \quad |z| \leqq 1,$$

that is,

$$\|[\varrho I - (\varrho-1)zT]h\| \geqq \|zTh\| \quad \text{for} \quad h \in \mathfrak{H}, \quad |z| \leqq 1,$$

which is equivalent to (2). The proof is complete.

1.3. Recall now a construction from [2]. Let T a power-bounded operator in $\mathscr{L}(\mathfrak{H})$. Put $\boldsymbol{H} = \bigoplus_{-\infty}^{\infty} \mathfrak{H}_k$, where each $\mathfrak{H}_k$ is a copy of $\mathfrak{H}$, and denote by $\{\overset{(k)}{h_k}\}_{k \in \mathbf{Z}}$ the elements of $\boldsymbol{H}$. We shall denote an element of the form $\{\dots, 0, \dots, \overset{(k-1)}{0}, \overset{(k)}{h}, \overset{(k+1)}{0}, \dots, 0, \dots\}$ simply by $\overset{(k)}{h}$. Let $\{p_k\}_{k \in \mathbf{Z}}$ be an arbitrary sequence of positive integers. Define $\boldsymbol{T} \in \mathscr{L}(\boldsymbol{H})$ as the operator $\overset{(k)}{h} \to \overset{(k+1)}{\overbrace{T^{p_k}h}}$. In [2] it is proved that if $T \in \mathscr{C}_\varrho$, then $\boldsymbol{T} \in \mathscr{C}_\varrho$.

We shall use also the following

1.4. Theorem. *If $T \in \mathscr{C}_\varrho$, the sequence $\{\|T^n h\|\}$ converges for all $h \in \mathfrak{H}$.*

For the proof, see [2] or [7].

§ 2. Weighted bilateral shifts

In this paragraph we shall consider a Hilbert space with an orthonormal basis $\{e_k\}_{k\in\mathbf{Z}}$ and corresponding *weighted* (bilateral) *shifts*, i.e. operators which transform e_k into $w_k e_{k+1}$, where $\{w_k\}_{k\in\mathbf{Z}}$ is a bounded sequence of complex numbers. Such a weighted shift is unitarily equivalent to the one with weights $\{|w_k|\}$ so we can suppose that the weights are nonnegative (see [5] or [6]).

We shall denote by $\{\dots, w_{-n}, \dots, w_{-1}, w_0, w_1, \dots, w_n, \dots\}$ or briefly by $\{w_k\}$ the weights as well as the operator itself.

2. 1. Proposition. *If* $\{w_k\}\in\mathscr{C}_\varrho$ *and* $\{s_k\}\in\mathscr{C}_\sigma$ *then* $\{w_k s_k\}\in\mathscr{C}_{\varrho\sigma}$.

Proof. Aplying 1. 1 we have that $\{w_k\}\otimes\{s_k\}\in\mathscr{C}_{\varrho\sigma}$. Notice that the subspace $\mathfrak{H}_0\subset\mathfrak{H}\otimes\mathfrak{H}$ generated by $\{e_k\otimes e_k\}_{k\in\mathbf{Z}}$ is invariant for $\{w_k\}\otimes\{s_k\}$ and the restriction to $\mathfrak{H}_0$ of this operator is also of class $\mathscr{C}_{\varrho\sigma}$. But this restriction is a weighted shift with weights $\{w_k s_k\}$.

2. 2. Corollary. *If* $\{w_k\}\in\mathscr{C}_\varrho$ *and* $0\leqq s_k\leqq w_k$, *then* $\{s_k\}\in\mathscr{C}_\varrho$.

Proof. One can find numbers $0\leqq\alpha_k\leqq 1$ such that $s_k=\alpha_k w_k$. Since $\{\alpha_k\}$ is a contraction, the conclusion follows.

2. 3 Proposition. $T=\{w_k\}\in\mathscr{C}_\varrho$ *if and only if*

(j) $$v(T)\leqq a=\min\{1,\varrho\},$$

(jj) $$\sum_{k=-\infty}^{\infty}[(\varrho-2)w_k^2+\varrho]x_k^2-\sum_{k=-\infty}^{\infty}2(\varrho-1)w_k x_k x_{k+1}\geqq 0$$

for every sequence of real numbers x_k *with* $\sum_{-\infty}^{\infty}x_k^2<\infty$.

Proof. Take $h=\sum z_k e_k$ and apply (1. 2). If we put $x_k=|z_k|$ we obtain (jj).

2. 4. Lemma. *The real infinite quadratic form*

$$\sum_{-\infty}^{\infty}a_k x_k^2-\sum_{-\infty}^{\infty}b_k x_k x_{k+1}\quad\left(\sum_{-\infty}^{\infty}x_k^2<\infty;\ a_k,\ b_k\ \textit{bounded}\right)$$

is positive semidefinite if and only if it can be written in the form

$$\sum(\alpha_k x_k-\beta_k x_{k+1})^2\quad(\alpha_k,\beta_k\in\mathbf{R}).$$

Proof. See [3].

2. 5. Theorem. $\{w_k\}\in\mathscr{C}_2$ *if and only if the weights are of the form* $w_k^2 = (1-c_k)(1+c_{k+1})$, $c_k\in[-1,1]$, $k\in\mathbf{Z}$.

Proof. As known, $\mathscr{C}_2$ consists of the operators T with $|(Th, h)| \leqq \|h\|^2$, that is, of numerical radius contradictions. Using 2. 3, a necessary and sufficient condition for $\{w_k\}$ to be of the class $\mathscr{C}_2$ is that

$$\sum_{-\infty}^{\infty} x_k^2 - \sum_{-\infty}^{\infty} w_k x_k x_{k+1} \geqq 0 \quad \text{if} \quad \sum_{-\infty}^{\infty} x_k^2 < \infty,$$

that is (using 2. 4)

$$\alpha_k^2 + \beta_{k-1}^2 = 1, \; 2\alpha_k \beta_k = w_k, \quad \text{where} \quad \alpha_k, \beta_k \in \mathbf{R}, \; k \in \mathbf{Z}.$$

We have $w_k^2 = 4\alpha_k^2 \beta_k^2 = 2\alpha_k^2(2-2\alpha_{k+1}^2)$. Put $c_k = 1-2\alpha_k^2$, and the conclusion follows.

2. 6. Proposition. *$T=\{w_k\} \in \mathscr{C}_\varrho$ $(\varrho > 2)$ if and only if*

$$\text{(k)} \quad \nu(T) \leqq 1, \qquad \text{(kk)} \quad \{u_k\} \in \mathscr{C}_2,$$

where

$$u_k = \frac{2(\varrho-1)w_k}{\sqrt{(\varrho-2)w_k^2+\varrho}\,\sqrt{(\varrho-2)w_{k+1}^2+\varrho}}.$$

Proof. Take $y_k^2 = [(\varrho-2)w_k^2+\varrho]x_k^2$ in (jj) of (2. 3).

2.7 Proposition. *If $T=\{w_k\} \in \mathscr{C}_\varrho$ then $\prod_{-\infty}^{\infty} w_k$ converges (possibly to 0).*

Observe that $\prod_{-\infty}^{\infty} w_k = \lim \|T^n e_0\| \cdot \lim \|T^{*n} e_0\|$; the limits i on the right hand side exist by 1. 4.

Observe that $\prod_{-\infty}^{\infty} w_k \neq 0$ implies $w_k \to 1$ as $k \to \pm\infty$.

2. 8. Proposition. *If $\{w_k\} \in \mathscr{C}_2$ then $\prod_{-\infty}^{\infty} w_k \leqq 1$.*

Indeed, from 2. 5 we have

$$\prod w_k = \prod \sqrt{(1-c_k)(1+c_{k+1})} = \prod \sqrt{1-c_k^2} \leqq 1.$$

2. 9. Definition. Let $\{w_k\}$ be a weighted shift. A *compression* of $\{w_k\}$ is any weighted shift obtained by substituting a finite sequence of consecutive weights by their product.

For example, $\{\dots, w_{-2}, w_{-1}w_0, w_1, \dots\}$ and $\{\dots, w_{-2}, w_{-1}w_0w_1, w_2, \dots\}$ are compressions of the shift $\{w_k\}$.

2. 10. Proposition. *Every compression of a weighted shift $\{w_k\}$ of class $\mathscr{C}_\varrho$ is also of class $\mathscr{C}_\varrho$.*

Proof. Choose $m \leqq n$ and let $\{v_k\}$ be the weighted shift with

$$v_k = w_k \quad \text{for} \quad k < m, \qquad v_m = w_m \dots w_n, \qquad v_k = w_{k+n-m} \quad \text{for} \quad k > m.$$

To prove that $\{v_k\} \in \mathscr{C}_\varrho$ we shall repeat the construction of 1. 3 by choosing $p_k = 1$ for $k \neq m$ and $p_m = n - m + 1$. Let $\mathfrak{H}_0$ be the subspace of $\boldsymbol{H}$ with base $\{\overset{(k)}{e_k}\}$ for $k \leqq m$, and $\{e^{(k)}_{k+n-m}\}$ for $k > m$. $\mathfrak{H}_0$ will be invariant for $\boldsymbol{T}$ and $\boldsymbol{T}|\mathfrak{H}_0$ will be just the weighted shift with weights $\{v_k\}$.

2. 11 Proposition. *If $\{w_k\} \in \mathscr{C}_\varrho$ then $a = \Pi w_k \leqq 1$.*

Proof. For $\varrho = 2$, (and then also for $\varrho < 2$) this is contained in 2. 8. Denote by T_n the weighted shift obtained from $T = \{w_k\}$ by compression of weights from w_{-n} to w_n. By 2. 10, $T_n \in \mathscr{C}_\varrho$. If $a > 0$, then $T_n \to \{\dots, 1, \dots, 1, a, 1, \dots, 1, \dots\}$ (strongly). It follows that $\{\dots, 1, a, 1, \dots\} \in \mathscr{C}_\varrho$. If $a > 1$, by Corollary 2. 2 we may suppose $1 < a < \dfrac{\varrho}{\varrho - 2}$. Using 2. 6 we deduce that

$$u_k = \left\{\dots, 1, \dots, 1, \sqrt{\frac{2(\varrho-1)}{(\varrho-2)a^2+\varrho}}, \sqrt{\frac{2(\varrho-1)}{(\varrho-2)a^2+\varrho}} \cdot a, 1, \dots\right\} \in \mathscr{C}_2.$$

But $1 \geqq \prod u_k = \dfrac{2(\varrho-1)a}{(\varrho-2)a^2+\varrho} > 1 \left(\text{since } a < \dfrac{\varrho}{\varrho-2}\right)$ which is impossible.

2. 12. Theorem. *If $\{w_k\} \in \mathscr{C}_\varrho$ and $\prod\limits_{-\infty}^{\infty} w_k = 1$, then $w_k = 1$ for every $k \in \mathbf{Z}$.*

Proof. We may suppose $\varrho > 2$. Suppose some w_k differ from 1. Then we find an m such that $\prod\limits_{-\infty}^{m} w_k = a \neq 1$. Compressing weights from w_{m-n} to w_m and taking $n \to \infty$ it follows that $\{\dots, 1, a, w_{m+1}, \dots\} \in \mathscr{C}_\varrho$. Compressing weights from w_{m+1} to w_{m+n} and passing to limit, we deduce $\{\dots, 1, a, a^{-1}, 1, \dots\} \in \mathscr{C}_\varrho$. Considering, if necessary, the adjoint shift we may assume that $a < 1$. Now using 2. 6 we obtain:

$$u_k = \left\{\dots, 1, \dots 1, \sqrt{\frac{2(\varrho-1)}{(\varrho-2)a^2+\varrho}}, \frac{2(\varrho-1)a^2}{\sqrt{(\varrho-2)a^2+\varrho}\,\sqrt{(\varrho-2)+\varrho a^2}}, \right.$$

$$\left. \sqrt{\frac{2(\varrho-1)}{(\varrho-2)+a^2}}, 1, \dots\right\} \in \mathscr{C}_2.$$

Using 2. 5 we deduce

$$u_k = (1-c_k)(1+c_{k+1}) = \begin{cases} 1 & \text{for } |k|>1, \\ \dfrac{2}{a^2+1-\varepsilon} & \text{for } k=-1, \\ \dfrac{4a^2}{(a^2+1)^2-\varepsilon^2} & \text{for } k=0, \\ \dfrac{2}{a^2+1+\varepsilon} & \text{for } k=1, \end{cases}$$

where we have put $\varepsilon = \dfrac{a^2-1}{\varrho-1}$. By the fact that $\mathscr{C}_\varrho$ is an increasing function of ϱ we may suppose $|\varepsilon|<1$. We have

$$1 = \prod_{-\infty}^{-2} u_k = (1+c_{-1}) \prod_{-\infty}^{-2} (1-c_k^2); \quad \text{hence} \quad c_{-1} \geqq 0.$$

By the same method, from $\prod_2^\infty u_k = 1$ it follows that $c_2 \leqq 0$. Then,

$$(1-c_{-1})(1+c_0) = \frac{2}{1+a^2-\varepsilon}, \quad (1-c_0)(1+c_1) = \frac{4a^2}{(1+a^2)^2-\varepsilon^2},$$

and

$$(1-c_1)(1+c_2) = \frac{2}{1+a^2+\varepsilon}.$$

From the first equality and from $c_{-1} \geqq 0$ we deduce

$$1-c_0 \leqq \frac{2a^2-2\varepsilon}{1+a^2-\varepsilon},$$

while from the last one and from $c_2 \leqq 0$ we have

$$1+c_1 \leqq \frac{2a^2+2\varepsilon}{1+a^2+\varepsilon}.$$

Hence,

$$\frac{2a^2-2\varepsilon}{1+a^2-\varepsilon} \cdot \frac{2a^2+2\varepsilon}{1+a^2+\varepsilon} \geqq (1-c_0)(1+c_1) = \frac{4a^2}{(1+a^2)^2-\varepsilon^2}$$

and it follows that $\varepsilon = \dfrac{a^2-1}{\varrho-1} = 0$, $a=1$, a contradiction. The proof is complete.

2. 13. Corollary. *If $T=\{w_k\}$ is invertible and $T \in \mathscr{C}_\varrho$, $T^{-1} \in \mathscr{C}_\varrho$, then T is unitary.*

Proof. It suffices to remark that T^{-1} is also a weighted shift with weights $\{w_k^{-1}\}$. Using 2. 11 we deduce $\prod w_k \leqq 1$ and $\prod (w_k^{-1}) \leqq 1$ hence $\prod w_k = 1$, that is (from 2. 12) $w_k = 1$ for every $k \in \mathbf{Z}$.

§ 3. Invertible operators of class $\mathscr{C}_\varrho$

Let $\mathfrak{H}$ be a Hilbert space and T an invertible operator of class $\mathscr{C}_\varrho$.

3. 1. Theorem. *If* $0 \neq h \in \mathfrak{H}$ *and* $w_k = \dfrac{\|T^{k+1}h\|}{\|T^k h\|}$ $(k \in \boldsymbol{Z})$ *then* $\{w_k\}$ *is a weighted shift of class* $\mathscr{C}_\varrho$.

Proof. We construct, as in 1. 3, the space $\boldsymbol{H}$ and the operator $\boldsymbol{T}$ with all $p_i = 1$. Put $\boldsymbol{h}_k = \overbrace{T^k h}^{(k)}$ $(k \in \boldsymbol{Z})$. Let $\mathfrak{H}_0$ be the subspace $\bigvee_{k=-\infty}^{\infty} \boldsymbol{h}_k$. Then $\mathfrak{H}_0$ has the orthonormal basis $e_k = \dfrac{\boldsymbol{h}_k}{\|\boldsymbol{h}_k\|}$.

It is easy to see that $\boldsymbol{T}$ leaves $\mathfrak{H}_0$ invariant, and $\boldsymbol{T}|_{\mathfrak{H}_0}$ is just the desired weighted shift. Using 1. 7 and 1. 2 the proof is complete.

3. 2. Corollary. *If* $T \in \mathscr{C}_\varrho$ *and* T *is invertible, then*

$$\lim \|T^n h\| \leqq \lim \|T^{-n} h\| \quad \textit{for} \quad h \in \mathfrak{H}.$$

Proof. Using 2. 11 and 3. 1 we have

$$1 \geqq \prod \|T^{k+1}h\| \cdot \|T^k h\|^{-1} = \frac{\lim \|T^n h\|}{\lim \|T^{-n} h\|}.$$

3. 3. Corollary. *If* $T \in \mathscr{C}_\varrho$, *$T$ is invertible, and* $\lim \|T^n h\| = \lim \|T^{-n} h\|$, *then*

$$\|T^n h\| = \|h\| \quad \textit{for} \quad n = 1, 2, \ldots.$$

Proof. Obvious from 2. 12 and 3. 1.

3. 4. Corollary. *If* $T \in \mathscr{C}_\varrho$ *and* $\lim \|T^n h\| = \lim \|T^{-n} h\|$ *for all* $h \in \mathfrak{H}$, *then T is unitary.*

3. 5. Corollary. (Stampfli [4].) *If* T, T^{-1} *are both of class* $\mathscr{C}_\varrho$, *then T is unitary.*

Proof. Obvious from 3. 2 and 3. 4.

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Erweiterung von Halbgruppen durch wiederholte Quotientenbildung. II

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Einleitung

Im Teil I dieser Arbeit [4] untersuchten wir Halbgruppenerweiterungen, die sich durch wiederholte Bildung von Rechtsquotientenhalbgruppen und Linksquotientenhalbgruppen einer Halbgruppe $\mathfrak{N}$ ergeben. Dabei verstehen wir unter einer Rechtsquotientenhalbgruppe $\mathfrak{S}=\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ einer Halbgruppe $\mathfrak{N}$ nach einer Unterhalbgruppe $\mathfrak{n}$ regulärer Elemente[1]) von $\mathfrak{N}$ eine solche Oberhalbgruppe von $\mathfrak{N}$ mit Einselement, in der jedes Element $\alpha \in \mathfrak{n}$ ein Inverses besitzt und deren Elemente sich als Rechtsquotienten $a\alpha^{-1}$ mit $a \in \mathfrak{N}$ und $\alpha \in \mathfrak{n}$ darstellen lassen. Eine solche Rechtsquotientenhalbgruppe $\mathfrak{S}=\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ existiert bekanntlich (vgl. [1] und [2]) genau dann, wenn die folgende Bedingung $Q_r(\mathfrak{N}, \mathfrak{n})$ erfüllt ist: Zu je zwei Elementen $a \in \mathfrak{N}$ und $\alpha \in \mathfrak{n}$ gibt es Elemente $l \in \mathfrak{N}$ und $\lambda \in \mathfrak{n}$ mit $a\lambda=\alpha l$. Im Falle ihrer Existenz ist dann die Rechtsquotientenhalbgruppe $\mathfrak{S}=\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ durch die Halbgruppe $\mathfrak{N}$ und die rechtsseitige Nennermenge $\mathfrak{n}$ bis auf Isomorphie eindeutig bestimmt (vgl. [1], [2]). Dagegen können verschiedene Unterhalbgruppen $\mathfrak{n}_i$ $(i \in I)$ von $\mathfrak{N}$ zur gleichen Rechtsquotientenhalbgruppe $\mathfrak{S}=\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n}_i)$ führen. Unter diesen ist dann genau eine relativ maximale Nennermenge $\mathfrak{n}$ (sie besteht aus allen in $\mathfrak{S}$ invertierbaren Elementen von $\mathfrak{N}$) dadurch ausgezeichnet, daß aus $a \cdot b \in \mathfrak{n}$ für beliebige reguläre Elemente a und b von $\mathfrak{N}$ stets $a \in \mathfrak{n}$ und $b \in \mathfrak{n}$ folgt (vgl. [2]).

Ist es nun möglich, eine Halbgruppe $\mathfrak{N}$ zuerst zu einer Rechtsquotientenhalbgruppe $\mathfrak{N}_1=\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n}_1)$ nach einer Unterhalbgruppe $\mathfrak{n}_1 \subseteqq \mathfrak{N}$ zu erweitern, diese Halbgruppe $\mathfrak{N}_1$ wieder zu einer Linksquotientenhalbgruppe $\mathfrak{N}_2=\mathfrak{Q}_l(\mathfrak{N}_1, \mathfrak{n}_2)$ nach einer Unterhalbgruppe $\mathfrak{n}_2 \subseteqq \mathfrak{N}_1$ u.s.f., so nennen wir eine auf diese Weise nach k Schritten entstehende Oberstruktur $\mathfrak{N}_k$ von $\mathfrak{N}$ eine k-te r-Quotientenhalbgruppe $\mathfrak{N}_k=$ $=\mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \dots, \mathfrak{n}_k)$, wobei $\mathfrak{n}_\varkappa$ $(\varkappa=1, 2, \dots, k)$ die jeweils im $\varkappa$-ten Schritt verwendete Nennermenge bezeichnet (vgl. [4], Def. 1). Entsprechend definieren wir eine k-te

[1]) Reguläre Elemente erfüllen die Kürzungsregeln.

l-Quotientenhalbgruppe $\mathfrak{Q}_l^k(\mathfrak{N}; \mathfrak{n}_1, \ldots, \mathfrak{n}_k)$ einer Halbgruppe $\mathfrak{N}$, wobei aber der erste Erweiterungsschritt eine Linksquotientenerweiterung ist. Wir wollen auch in dem vorliegenden Teil II unsere Verabredung beibehalten, daß mit jeder Begriffsbildung bzw. Aussage auch die durch formale „Vertauschung von rechts und links" hervorgehende duale Begriffsbildung bzw. Aussage als gegeben angesehen wird.

Wir zeigten dann, daß wir für die bei der schrittweisen Quotientenerweiterung von $\mathfrak{N}$ verwendeten Nennermengen ohne Beschränkung der Allgemeinheit stets $\mathfrak{n}_{\varkappa-1} \subseteqq \mathfrak{n}_\varkappa$ und $\mathfrak{n}_{\varkappa-1}^{-1} \subseteqq \mathfrak{n}_\varkappa$ $(\varkappa=2, 3, \ldots, k)$ voraussetzen dürfen und betrachteten ihre Durchschnitte $\mathfrak{x}_\varkappa = \mathfrak{n}_\varkappa \cap \mathfrak{N}$ mit der Halbgruppe $\mathfrak{N}$. Eine so entstehende Unterhalbgruppenkette $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ nannten wir eine Q_r-Kette von $\mathfrak{N}$ der Länge k und wiesen nach, daß schon diese Unterhalbgruppen $\mathfrak{x}_\varkappa$ von $\mathfrak{N}$ die k-te r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \ldots, \mathfrak{n}_k)$ bis auf Isomorphie eindeutig festlegen, woraufhin wir die Bezeichnung $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N};, \mathfrak{x}_1, \ldots, \mathfrak{x}_k)$ einführten. Vor allen Dingen ist es nun möglich, notwendige und hinreichende Bedingungen für die Existenz der k-ten r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \ldots, \mathfrak{x}_k)$ auszusprechen (vgl. [4], Satz 3): Eine Kette $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ von Unterhalbgruppen regulärer Elemente einer Halbgruppe $\mathfrak{N}$ ist genau dann Q_r-Kette der Länge k, wenn die folgenden Forderungen erfüllt sind. 1) Für jedes $\varkappa=2, 3, \ldots, k$ und beliebige Elemente a und b von $\mathfrak{N}$ folgt aus $a \cdot b \in \mathfrak{x}_\varkappa$ und $b \in \mathfrak{x}_{\varkappa-1}$ (bzw. $a \in \mathfrak{x}_{\varkappa-1}$) stets $a \in \mathfrak{x}_\varkappa$ (bzw. $b \in \mathfrak{x}_\varkappa$). 2) Für jedes $\varkappa=1, 2, \ldots, k$ gilt die mit $Q_r^\varkappa(\mathfrak{N}; \mathfrak{x}_1, \ldots, \mathfrak{x}_\varkappa)$ bezeichnete Bedingung: Zu beliebigen Elementen $a^1 \in \mathfrak{N}$ und $x_\varkappa^1 \in \mathfrak{x}_\varkappa$ gibt es geeignete Elemente[2])

$$\begin{aligned} &a^j \in \mathfrak{N}, \quad x_\varkappa^j \in \mathfrak{x}_\varkappa &&(j=2, 3, \ldots, \varkappa),\\ &b^j \in \mathfrak{N}, \quad y_\varkappa^j \in \mathfrak{x}_\varkappa &&(j=1, 2, \ldots, \varkappa),\\ &u_j \text{ und } v_j \text{ aus } \mathfrak{x}_j &&(j=1, 2, \ldots, \varkappa),\end{aligned}$$

so daß die folgenden Gleichungen erfüllt sind

$$a^\lambda u_\lambda = v_\lambda b^\lambda, \quad x_\varkappa^\lambda u_\lambda = v_\lambda y_\varkappa^\lambda \qquad (\lambda=1, 2, \ldots, \varkappa),$$

$$u_\varkappa = y_\varkappa^\varkappa, \quad v_\varkappa = x_\varkappa^\varkappa,$$

$$a^{2i} = a^{2i+1}, \quad x_\varkappa^{2i} = x_\varkappa^{2i+1}, \quad b^{2i-1} = b^{2i}, \quad y_\varkappa^{2i-1} = y_\varkappa^{2i},$$

wobei der Index i für ungerades $\varkappa$ die Werte $1, 2, \ldots, \frac{\varkappa-1}{2}$ und für gerades $\varkappa$ die Werte $1, 2, \ldots, \frac{\varkappa}{2}$ durchläuft.

Für die duale Bedingung $Q_l^\varkappa(\mathfrak{N}; \mathfrak{x}_1, \ldots, \mathfrak{x}_\varkappa)$ sind natürlich die Faktoren in allen auftretenden Produkten zu vertauschen.

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[2]) Die hochgestellten Indizes dienen zur Unterscheidung der Elemente.

§ 4

Freilich ist mit diesem allgemeinen Kriterium noch nichts darüber ausgesagt, ob es für eine natürliche Zahl k überhaupt eine Halbgruppe $\mathfrak{N}$ gibt, welche eine k-te r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \ldots, \mathfrak{x}_k)$ zu bilden gestattet, ohne daß man dabei mit weniger als k Erweiterungsschritten auskommen kann. Wir ergänzen daher unsere Untersuchungen aus Teil I durch folgenden

SATZ 6 (Hauptsatz). *Für jede natürliche Zahl k gibt es eine Halbgruppe $\mathfrak{N}$, zu der eine k-te r-Quotientenhalbgruppe $\mathfrak{N}_k = \mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{x}_1, \ldots, \mathfrak{x}_k)$ existiert, die auf keine Weise in weniger als k Schritten durch Quotientenerweiterung gewonnen werden kann.*

Wir werden diesen Satz zusammen mit seiner dualen Aussage beweisen, indem wir für jede natürliche Zahl $k \geqq 1$ zwei (zueinander duale) Halbgruppen $\mathfrak{F}_k$ und $\mathfrak{H}_k$ angeben, die folgende Eigenschaften besitzen:

I) *In $\mathfrak{F}_k$ gibt es eine Q_r-Kette der Länge k und eine Q_l-Kette der Länge $k+1$, die beide mit $\mathfrak{F}_k$ selbst enden.*

In $\mathfrak{H}_k$ gibt es eine Q_l-Kette der Länge k und eine Q_r-Kette der Länge $k+1$, die beide mit $\mathfrak{H}_k$ selbst enden.

II) *Jede mit $\mathfrak{F}_k$ endende Q_r-Kette bzw. Q_l-Kette von $\mathfrak{F}_k$ hat mindestens die Länge k bzw. $k+1$.*

Jede mit $\mathfrak{H}_k$ endende Q_l-Kette bzw. Q_r-Kette von $\mathfrak{H}_k$ hat mindestens die Länge k bzw. $k+1$.

Das heißt: Die Halbgruppen $\mathfrak{F}_k$ und $\mathfrak{H}_k$ sind in Gruppen einbettbar, die durch Quotientenbildung nicht in weniger als k Schritten gewonnen werden können.

Wir konstruieren zunächst gewisse Halbgruppen $\mathfrak{F}_k$ und $\mathfrak{H}_k$, von denen wir dann in den beiden folgenden Paragraphen zeigen werden, daß sie die in I) und II) genannten Eigenschaften besitzen. Wir zeigen für jede natürliche Zahl k die Existenz zweier Halbgruppen $\mathfrak{F}_k$ und $\mathfrak{H}_k$ mit den Eigenschaften:

a) *$\mathfrak{F}_k$ und $\mathfrak{H}_k$ sind zueinander antiisomorphe reguläre Halbgruppen mit dem gleichen Einselement.*

b) *Es gibt einen Homomorphismus i_k von $\mathfrak{F}_k$ und einen Homomorphismus j_k von $\mathfrak{H}_k$ in (für $k \geqq 1$ sogar auf) die additive Halbgruppe der nichtnegativen ganzen Zahlen.*

c) *Der Antiisomorphismus ψ_k von $\mathfrak{F}_k$ auf $\mathfrak{H}_k$ erfüllt bezüglich der in* b) *genannten Homomorphismen i_k und j_k*

$$i_k(f) = j_k(\psi_k(f)) \quad \textit{für alle} \quad f \in \mathfrak{F}_k,$$

womit auch $j_k(h) = i_k(\psi_k^{-1}(h))$ für alle $h \in \mathfrak{H}_k$ gilt.

Die Halbgruppen $\mathfrak{F}_k$ und $\mathfrak{H}_k$ konstruieren wir induktiv, wobei wir für $k=0$ die aus nur einem Element e bestehende Halbgruppe $\mathfrak{F}_0=\mathfrak{H}_0$ mit $i_0(e)=j_0(e)=0$ und der identischen Abbildung ψ_0 verwenden. Es seien nun $\mathfrak{F}_k$ und $\mathfrak{H}_k$ Halbgruppen mit den Eigenschaften a), b) und c). Dann definieren wir zunächst $\mathfrak{F}_{k+1}$ als diejenige Halbgruppe, die von den Elementen von $\mathfrak{H}_k$ und zwei weiteren Elementen $C=C_{k+1}$ und $D=D_{k+1}$ erzeugt wird[3], wobei neben den Relationen von $\mathfrak{H}_k$ noch folgende Relationen gefordert werden:

(1) $$D_{k+1}C_{k+1} = C_{k+1}D_{k+1}^2,$$

(2) $$hC_{k+1} = C_{k+1}h, \quad D_{k+1}h = hD_{k+1}^{2^{j_k(h)}} \quad \text{für alle} \quad h\in\mathfrak{H}_k,$$

(3) $$eC_{k+1} = C_{k+1}, \quad eD_{k+1} = D_{k+1} \quad \text{für das Einselement } e \text{ von } \mathfrak{H}_k.$$

Ersichtlich ist das Einselement e von $\mathfrak{H}_k$ wegen $j_k(e)=0$ auch Einselement von $\mathfrak{F}_{k+1}$. Die Elemente von $\mathfrak{F}_{k+1}$ sind dann jedenfalls von der Form hC^cD^d mit $h\in\mathfrak{H}_k$ und nichtnegativen ganzen Zahlen c und d, und je zwei Elemente von $\mathfrak{F}_{k+1}$ werden gemäß

(*) $$h_1C^{c_1}D^{d_1}\cdot h_2C^{c_2}D^{d_2}=h_1h_2C^{c_1+c_2}D^{d_2+2^{j_k(h_2)+c_2}d_1}$$

multipliziert. Darüber hinaus besitzt jedes Element von $\mathfrak{F}_{k+1}$ sogar genau eine solche Darstellung hC^cD^d. Dazu betrachte man die Produktmenge $F_{k+1} = \mathfrak{H}_k\times\times N_0\times N_0$, worin N_0 die Menge der nichtnegativen ganzen Zahlen bezeichnet, und definiere in F_{k+1}:

α) Es sei $(h_1, c_1, d_1)=(h_2, c_2, d_2)$ genau dann, wenn $h_1=h_2$, $c_1=c_2$ und $d_1=d_2$ gilt;

β) $$(h_1, c_1, d_1)\cdot(h_2,c_2,d_2) = (h_1h_2, c_1+c_2, d_2+2^{j_k(h_2)+c_2}d_1).$$

Wegen α) wird durch β) ersichtlich eine assoziative Multiplikation in F_{k+1} definiert, so daß in F_{k+1} auch die Relationen (1), (2) und (3) erfüllt sind. Wir zeigen noch, daß $\mathfrak{F}_{k+1}$ eine reguläre Halbgruppe ist. Dazu sei

$$hC^cD^d\cdot h_1C^{c_1}D^{d_1}=hC^cD^d\cdot h_2C^{c_2}D^{d_2},$$

also

$$hh_1C^{c+c_1}D^{d_1+2^{j_k(h_1)+c_1}d} = hh_2C^{c+c_2}D^{d_2+2^{j_k(h_2)+c_2}d},$$

woraus sich zunächst

$$hh_1 = hh_2, \quad c+c_1 = c+c_2, \quad d_1+2^{j_k(h_1)+c_1}d = d_2+2^{j_k(h_2)+c_2}d$$

ergibt. Wegen der Regularität von $\mathfrak{H}_k$ gilt $h_1=h_2$, außerdem folgt $c_1=c_2$ und aus beidem $d_1=d_2$, womit

$$h_1C^{c_1}D^{d_1}=h_2C^{c_2}D^{d_2}$$

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[3]) Sind keine Verwechslungen zu befürchten, werden wir zur Vereinfachung im Folgenden auf die Indizes verzichten und einfach C und D schreiben.

gezeigt ist. In gleicher Weise zeigt man die Gültigkeit der rechtsseitigen Kürzungsregel. Es ist also $\mathfrak{F}_{k+1}$ eine reguläre Oberhalbgruppe von $\mathfrak{H}_k$ mit demselben Einselement. Wir betrachten nun die zur Halbgruppe $\mathfrak{F}_{k+1}$ duale Halbgruppe $\mathfrak{F}_{k+1}^{\circ} = \mathfrak{H}_{k+1}$, die also dadurch entsteht, daß man in $\mathfrak{F}_{k+1}$ die Multiplikation ab durch die duale Operation $a \circ b = ba$ ersetzt. Da wir $\mathfrak{F}_k$ und $\mathfrak{H}_k$ als duale Halbgruppen vorausgesetzt hatten, wobei es einen Antiisomorphismus ψ_k von $\mathfrak{F}_k$ auf $\mathfrak{H}_k$ gibt, der c) erfüllt, ist die Unterhalbgruppe $\mathfrak{H}_k^{\circ}$ von $\mathfrak{F}_{k+1}^{\circ} = \mathfrak{H}_{k+1}$ isomorph zu $\mathfrak{F}_k$, weshalb wir $\mathfrak{H}_{k+1}$ als Oberhalbgruppe von $\mathfrak{F}_k$ annehmen dürfen. Setzen wir nun noch $C_{k+1}^{\circ} = B_{k+1}$ und $D_{k+1}^{\circ} = A_{k+1}$, so wird durch

$$\psi_{k+1}(hC_{k+1}^{c_{k+1}} D_{k+1}^{d_{k+1}}) = A_{k+1}^{d_{k+1}} B_{k+1}^{c_{k+1}} \psi_k^{-1}(h)$$

für alle Elemente $hC_{k+1}^{c_{k+1}} D_{k+1}^{d_{k+1}} \in \mathfrak{F}_{k+1}$ ein Antiisomorphismus ψ_{k+1} von $\mathfrak{F}_{k+1}$ auf $\mathfrak{H}_{k+1}$ erklärt, der Fortsetzung von ψ_k^{-1} ist. Mit anderen Worten: Wir betrachten $\mathfrak{H}_{k+1}$ als die von der Halbgruppe $\mathfrak{F}_k = \psi_k^{-1}(\mathfrak{H}_k)$ und zwei weiteren Elementen $A_{k+1} = \psi_{k+1}(D_{k+1})$ und $B_{k+1} = \psi_{k+1}(C_{k+1})$ erzeugte Halbgruppe, wobei zu den Relationen von $\mathfrak{F}_k$ noch die folgenden zu (1), (2) und (3) dualen Relationen hinzugenommen werden:

(1′) $$B_{k+1} A_{k+1} = \psi_{k+1}(D_{k+1} C_{k+1}) = \psi_{k+1}(C_{k+1} D_{k+1}^2) = A_{k+1}^2 B_{k+1},$$

(2′₁) $$B_{k+1} f = \psi_{k+1}(hC_{k+1}) = \psi_{k+1}(C_{k+1} h) = f B_{k+1}$$

sowie

(2′₂) $$f A_{k+1} = \psi_{k+1}(D_{k+1} h) = \psi_{k+1}(h D_{k+1}^{2^{j_k(h)}}) = A_{k+1}^{2^{j_k(\psi_k(f))}} f = A_{k+1}^{2^{i_k(f)}} f$$

für alle Elemente $f = \psi_k^{-1}(h)$ aus $\mathfrak{F}_k$,

(3′) $$B_{k+1} e = \psi_{k+1}(eC_{k+1}) = B_{k+1}$$

$$A_{k+1} e = \psi_{k+1}(eD_{k+1}) = A_{k+1}$$

für das Einselement $e = \psi_k^{-1}(e)$ von $\mathfrak{F}_k$.

Damit ist $\mathfrak{H}_{k+1}$ eine reguläre Oberhalbgruppe von $\mathfrak{F}_k$ mit demselben Einselement, deren Elemente eindeutig in der Form $A_{k+1}^{a_{k+1}} B_{k+1}^{b_{k+1}} f$ mit $f \in \mathfrak{F}_k$ und nichtnegativen ganzen Zahlen a_{k+1} und b_{k+1} dargestellt werden können. Je zwei Elemente von $\mathfrak{H}_{k+1}$ werden nach der Regel

(**) $$A^{a_1} B^{b_1} f_1 \cdot A^{a_2} B^{b_2} f_2 = A^{a_1 + 2^{i_k(f_1)+b_1} a_2} B^{b_1+b_2} f_1 f_2$$

multipliziert, wenn wir auch hier zur Vereinfachung auf die Indizes $k+1$ verzichten und einfach A bzw. B schreiben. Um noch die Gültigkeit von b) und c) für die Halbgruppen $\mathfrak{F}_{k+1}$ und $\mathfrak{H}_{k+1}$ zu zeigen, definieren wir

$$i_{k+1}(hC^c D^d) = j_k(h) + c \quad \text{für alle Elemente} \quad hC^c D^d \in \mathfrak{F}_{k+1},$$

$$j_{k+1}(A^a B^b f) = i_k(f) + b \quad \text{für alle Elemente} \quad A^a B^b f \in \mathfrak{H}_{k+1}.$$

Ersichtlich handelt es sich wegen (*) und (**) um Homomorphismen von $\mathfrak{F}_{k+1}$ bzw. von $\mathfrak{H}_{k+1}$ auf die additive Halbgruppe der nichtnegativen ganzen Zahlen. Es sei noch bemerkt, daß sich die Multiplikation in $\mathfrak{F}_{k+1}$ bzw. in $\mathfrak{H}_{k+1}$ dann auch gemäß

$$h_1 C^{c_1} D^{d_1} \cdot h_2 C^{c_2} D^{d_2} = h_1 h_2 C^{c_1+c_2} D^{d_2+2^{i_{k+1}(h_2 C^{c_2} D^{d_2})} d_1}$$

bzw.

$$A^{a_1} B^{b_1} f_1 \cdot A^{a_2} B^{b_2} f_2 = A^{a_1+2^{j_{k+1}(A^{a_1} B^{b_1} f_1)} a_2} B^{b_1+b_2} f_1 f_2$$

kennzeichnen läßt. Schließlich erfüllt ψ_{k+1} auch die Forderung von c).

Es ist also etwa $\mathfrak{F}_1$ eine reguläre Halbgruppe mit Einselement, deren Elemente sich eindeutig in der Form

$$e C_1^{c_1} D_1^{d_1} = C_1^{c_1} D_1^{d_1} \quad (c_1, d_1 \text{ nichtnegative ganze Zahlen})$$

schreiben lassen, wobei $i_1(C_1^{c_1} D_1^{d_1}) = c_1$ für jedes Element von $\mathfrak{F}_1$ gilt und die Multiplikation in $\mathfrak{F}_1$ nach der Regel

$$C_1^{c_1} D_1^{d_1} \cdot C_1^{c_1'} D_1^{d_1'} = C_1^{c_1+c_1'} D_1^{d_1'+2^{c_1'} d_1}$$

erfolgt. Wir bemerken noch, daß diese Halbgruppen $\mathfrak{F}_1$ und entsprechend $\mathfrak{H}_1$ gerade die bereits in § 1 von Teil I verwendeten Halbgruppen $\mathfrak{F}$ und $\mathfrak{H}$ sind.

Allgemein ist $\mathfrak{F}_k$ eine reguläre Oberhalbgruppe von $\mathfrak{H}_{k-1}$ mit demselben Einselement, deren Elemente sich eindeutig in der Form

$$A_{k-1}^{a_{k+1}} B_{k-1}^{b_{k-1}} A_{k-3}^{a_{k-3}} B_{k-3}^{b_{k-3}} \ldots A_2^{a_2} B_2^{b_2} C_1^{c_1} D_1^{d_1} C_3^{c_3} D_3^{d_3} \ldots C_k^{c_k} D_k^{d_k}$$

für ungerades k bzw.

$$A_{k-1}^{a_{k-1}} B_{k-1}^{b_{k-1}} A_{k-3}^{a_{k-3}} B_{k-3}^{b_{k-3}} \ldots A_1^{a_1} B_1^{b_1} C_2^{c_2} D_2^{d_2} C_4^{c_4} D_4^{d_4} \ldots C_k^{c_k} D_k^{d_k}$$

für gerades k mit nichtnegativen ganzen Zahlen als Exponenten darstellen lassen. Dabei gilt für alle Elemente $f_k \in \mathfrak{F}_k$:

$$i_k(f_k) = c_1 + b_2 + c_3 + b_4 + \cdots + b_{k-1} + c_k, \quad k \text{ ungerade}$$

bzw.

$$i_k(f_k) = b_1 + c_2 + b_3 + c_4 + \cdots + b_{k-1} + c_k, \quad k \text{ gerade.}$$

Entsprechende Darstellungen gelten für die Halbgruppe $\mathfrak{H}_k$.

§ 5

Wir wenden uns jetzt der Untersuchung von Q_r-Ketten und Q_l-Ketten der im vorigen Paragraphen konstruierten Halbgruppen $\mathfrak{F}_k$ bzw. $\mathfrak{H}_k$ für $k \geqq 1$ zu. Auf Grund der Dualität von $\mathfrak{F}_k$ und $\mathfrak{H}_k$ bilden Unterhalbgruppen

$$\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_l$$

von $\mathfrak{F}_k$ genau dann eine Q_r-Kette der Länge l von $\mathfrak{F}_k$, wenn die Unterhalbgruppen

$$\psi_k(\mathfrak{x}_1) \subseteqq \psi_k(\mathfrak{x}_2) \subseteqq \cdots \subseteqq \psi_k(\mathfrak{x}_l)$$

von $\mathfrak{H}_k$ eine Q_l-Kette von $\mathfrak{H}_k$ der gleichen Länge darstellen. Für den Nachweis der Behauptung I) im Anschluß an Satz 6 stellen wir zunächst zwei Hilfssätze bereit, die übrigens auch unter etwas allgemeineren Voraussetzungen erfüllt sind.

Hilfssatz 5. *Ist $\mathfrak{M}$ eine reguläre Halbgruppe mit Einselement, die sich als Komplexprodukt $\mathfrak{M}=\mathfrak{N}\mathfrak{G}=\mathfrak{G}\mathfrak{N}$ einer Unterhalbgruppe $\mathfrak{N}\subseteqq\mathfrak{M}$ und einer Untergruppe $\mathfrak{G}\subseteqq\mathfrak{M}$ darstellen läßt, wobei $a\mathfrak{G}=\mathfrak{G}a$ für alle Elemente $a\in\mathfrak{N}$ gilt, dann existiert mit jeder Rechtsquotientenhalbgruppe $\mathfrak{S}=\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ von $\mathfrak{N}$ nach einer Unterhalbgruppe $\mathfrak{n}$ von $\mathfrak{N}$ auch die Rechtsquotientenhalbgruppe $\mathfrak{T}=\mathfrak{Q}_r(\mathfrak{N}\mathfrak{G}, \mathfrak{n}\mathfrak{G})$ von $\mathfrak{M}=\mathfrak{N}\mathfrak{G}$ nach der rechtsseitigen Nennermenge $\mathfrak{n}\mathfrak{G}\subseteqq\mathfrak{M}$. Überdies gewinnt man $\mathfrak{T}=\mathfrak{Q}_r(\mathfrak{N}\mathfrak{G}, \mathfrak{n}\mathfrak{G})$ als Komplexprodukt $\mathfrak{T}=\mathfrak{S}\mathfrak{G}$ von $\mathfrak{S}=\mathfrak{Q}_r(\mathfrak{N}, \mathfrak{n})$ mit $\mathfrak{G}$, und für alle Elemente $s\in\mathfrak{S}$ gilt $s\mathfrak{G}=\mathfrak{G}s$.*

Da gemäß [2] jede Quotientenhalbgruppe einer regulären Halbgruppe ebenfalls regulär ist, folgt durch vollständige Induktion über k aus Hilfssatz 5 und seiner dualen Aussage der

Hilfssatz 6. *Ist $\mathfrak{M}$ eine reguläre Halbgruppe mit Einselement, die sich als Komplexprodukt $\mathfrak{M}=\mathfrak{N}\mathfrak{G}=\mathfrak{G}\mathfrak{N}$ einer Unterhalbgruppe $\mathfrak{N}\subseteqq\mathfrak{M}$ und einer Untergruppe $\mathfrak{G}\subseteqq\mathfrak{M}$ darstellen läßt, wobei $a\mathfrak{G}=\mathfrak{G}a$ für alle Elemente $a\in\mathfrak{N}$ gilt, dann existiert mit jeder k-ten r-Quotientenhalbgruppe $\mathfrak{N}_k=\mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \ldots, \mathfrak{n}_k)$ von $\mathfrak{N}$ auch die k-te r-Quotientenhalbgruppe $\mathfrak{Q}_r^k(\mathfrak{N}\mathfrak{G}; \mathfrak{n}_1\mathfrak{G}, \ldots, \mathfrak{n}_k\mathfrak{G})$ von $\mathfrak{M}=\mathfrak{N}\mathfrak{G}$, die man als Komplexprodukt $\mathfrak{N}_k\mathfrak{G}$ von $\mathfrak{N}_k=\mathfrak{Q}_r^k(\mathfrak{N}; \mathfrak{n}_1, \ldots, \mathfrak{n}_k)$ mit $\mathfrak{G}$ gewinnen kann.*

Den ersten Teil der Behauptung I), daß es in der Halbgruppe $\mathfrak{F}_k$ $(k\geqq 1)$ eine Q_r-Kette $\mathfrak{x}_1\subseteqq\mathfrak{x}_2\subseteqq\cdots\subseteqq\mathfrak{x}_k=\mathfrak{F}_k$ gibt, verschärfen wir dahingehend, daß dabei stets das Einselement e von $\mathfrak{F}_k$ in $\mathfrak{x}_1$ enthalten ist, und beweisen dies zusammen mit der dualen Aussage über $\mathfrak{H}_k$ durch vollständige Induktion über k. Dabei können wir uns immer auf die Durchführung jeweils eines der zueinander dualen Schlüsse beschränken. Für $k=1$ zeigen wir, daß $\mathfrak{F}_1$ eine rechtsseitige Nennermenge von sich selbst ist, was wir zwar aus unseren Überlegungen in § 1 übernehmen könnten, hier aber der Vollständigkeit halber noch einmal ausführen wollen. Für beliebig vorgegebene Elemente $a=C_1^c D_1^d$ und $\alpha=C_1^\gamma D_1^\delta$ aus $\mathfrak{F}_1$ erfüllen die Elemente $l=C_1^c D_1^{2^\gamma d}$ und $\lambda=C_1^\gamma D_1^{2^c\delta}$ aus $\mathfrak{F}_1$ gerade die Gleichung

$$a\lambda = C_1^c D_1^d C_1^\gamma D_1^{2^c\delta} = X_1^{c+\gamma} D_1^{2^c\delta+2^\gamma p} = C_1^\gamma D_1^\delta C_1^c D_1^{2^\gamma d} = \alpha l,$$

womit $Q_r(\mathfrak{F}_1, \mathfrak{F}_1)$ nachgewiesen ist. Für den Induktionsschritt sei nun

$$\mathfrak{x}_1\subseteqq\mathfrak{x}_2\subseteqq\cdots\subseteqq\mathfrak{x}_k=\mathfrak{F}_k$$

eine Q_r-Kette der Länge k von $\mathfrak{F}_k$, wobei $\mathfrak{x}_1$ das Einselement e von $\mathfrak{F}_k$ enthält, also

$$\mathfrak{y}_1 \subseteqq \mathfrak{y}_2 \subseteqq \dots \subseteqq \mathfrak{y}_k = \mathfrak{H}_k \quad \text{mit} \quad \mathfrak{y}_\varkappa = \psi_k(\mathfrak{y}_\varkappa), \qquad \varkappa = 1, 2, \dots, k$$

eine Q_l-Kette der Länge k von $\mathfrak{H}_k$, wobei $\mathfrak{y}_1$ das Einselement von $\mathfrak{H}_k$ enthält. Nun ist die von $C = C_{k+1} \in \mathfrak{F}_{k+1}$ und $D = D_{k+1} \in \mathfrak{F}_{k+1}$ erzeugte Unterhalbgruppe mit Einselement $\mathfrak{z}_0$ der Elemente $C^c D^d$ von $\mathfrak{F}_{k+1}$ auch rechtsseitige Nennermenge von $\mathfrak{F}_{k+1}$, denn mit beliebigen Elementen $a = hC^c D^d \in \mathfrak{F}_{k+1}$ und $\alpha = C^\gamma D^\delta \in \mathfrak{z}_0$ erfüllen die Elemente $l = hC^c D^{2\gamma d} \in \mathfrak{F}_{k+1}$ und $\lambda = C^\gamma D^{2^{jk(h)}+c\delta} \in \mathfrak{z}_0$ die Gleichung

$$a\lambda = hC^c D^d \cdot C^\gamma D^{2^{jk(h)}+c\delta} = hC^{c+\gamma} D^{2^{jk(h)}+c\delta+2\gamma d} = C^\gamma D^\delta \cdot hC^c D^{2\gamma d} = \alpha l,$$

also gilt die Bedingung $Q_r(\mathfrak{F}_{k+1}, \mathfrak{z}_0)$. Die Elemente der Rechtsquotientenhalbgruppe $\mathfrak{Q}_r(\mathfrak{F}_{k+1}, \mathfrak{z}_0)$ gestatten die Darstellung

$$hC^c D^d D^{-\delta} C^{-\gamma} \quad (c, d, \gamma, \delta \text{ nichtnegative ganze Zahlen}),$$

und alle Elemente der Form $C^c D^d D^{-\delta} C^{-\gamma}$ bilden auf Grund der Relationen (1), (2) und (3) eine Untergruppe $\mathfrak{G}$ von $\mathfrak{Q}_r(\mathfrak{F}_{k+1}, \mathfrak{z}_0) = \mathfrak{H}_k \mathfrak{G}$. Es zeigt sich, daß sogar $h\mathfrak{G} = \mathfrak{G}h$ für alle Elemente $h \in \mathfrak{H}_k$ erfüllt ist. Da nach Induktionsvoraussetzung

$$\mathfrak{y}_1 \subseteqq \mathfrak{y}_2 \subseteqq \cdots \subseteqq \mathfrak{y}_k = \mathfrak{H}_k$$

eine Q_l-Kette von $\mathfrak{H}_k$ ist, erfüllt die Halbgruppe $\mathfrak{H}_k \mathfrak{G} = \mathfrak{Q}_r(\mathfrak{F}_{k+1}, \mathfrak{z}_0)$ alle Voraussetzungen der dualen Aussage von Hilfssatz 6. Es existiert also mit der k-ten l-Quotientenhalbgruppe

$$\mathfrak{S} = \mathfrak{Q}_l^k(\mathfrak{H}_k; \mathfrak{y}_1, \dots, \mathfrak{y}_k) = \mathfrak{Q}_l^k(\mathfrak{H}_k; \mathfrak{n}_1, \dots, \mathfrak{n}_k)$$

von $\mathfrak{H}_k$, wobei $\mathfrak{y}_\varkappa = \mathfrak{n}_\varkappa \cap \mathfrak{H}_k$ gilt, eine k-te l-Quotientenhalbgruppe $\mathfrak{T} = \mathfrak{S}\mathfrak{G} = \mathfrak{Q}_l^k(\mathfrak{H}_k \mathfrak{G}; n_1 \mathfrak{G}, \dots, \mathfrak{n}_k \mathfrak{G})$ von $\mathfrak{H}_k \mathfrak{G} = \mathfrak{Q}_r(\mathfrak{F}_{k+1}, \mathfrak{z}_0)$. Diese ist zugleich eine $(k+1)$-te r-Quotientenhalbgruppe $\mathfrak{T} = \mathfrak{Q}_r^{k+1}(\mathfrak{F}_{k+1}; \mathfrak{z}_0, \mathfrak{n}_1 \mathfrak{G}, \dots, \mathfrak{n}_k \mathfrak{G})$ von $\mathfrak{F}_{k+1}$, weshalb $\mathfrak{z}_0 \subseteqq \mathfrak{z}_1 \subseteqq \cdots \subseteqq \mathfrak{z}_k$ mit $\mathfrak{z}_\varkappa = \mathfrak{n}_\varkappa \mathfrak{G} \cap \mathfrak{F}_{k+1}$ für $\varkappa = 1, 2, \dots, k$ eine Q_r-Kette von $\mathfrak{F}_{k+1}$ der Länge $k+1$ ist, die mit der das Einselement von $\mathfrak{F}_{k+1}$ enthaltenden Unterhalbgruppe $\mathfrak{z}_0 \subseteqq \mathfrak{F}_{k+1}$ beginnt und mit $\mathfrak{z}_k = \mathfrak{F}_{k+1}$ endet. Aus $\mathfrak{z}_k = \mathfrak{n}_k \mathfrak{G} \cap \mathfrak{F}_{k+1}$ folgt nämlich $\mathfrak{H}_k = \mathfrak{y}_k = \mathfrak{n}_k \cap \mathfrak{H}_k \subseteqq \mathfrak{n}_k \mathfrak{G} \cap \mathfrak{F}_{k+1} = \mathfrak{z}_k$, was mit $\mathfrak{z}_0 \subseteqq \mathfrak{z}_k$ auf $\mathfrak{F}_{k+1} = \mathfrak{H}_k \mathfrak{z}_0 \subseteqq \mathfrak{z}_k$ führt. Damit ist die angegebene Verschärfung der ersten Teilaussage von I) bewiesen. Aus ihr erhält man sofort auch die zweite Teilaussage, indem man bei der schrittweisen Erweiterung von $\mathfrak{F}_k$ zu $\mathfrak{Q}_r^k(\mathfrak{F}_k; \mathfrak{x}_1, \dots, \mathfrak{x}_k)$ vor die erste Rechtsquotientenerweiterung eine triviale Linksquotientenerweiterung mit der Nennermenge $\mathfrak{x}_0 = \{e\} \subseteqq \mathfrak{x}_1$ davorsetzt. Damit sind die Aussagen von I) für alle Halbgruppen $\mathfrak{F}_k$ und $\mathfrak{H}_k$ $(k \geqq 1)$ nachgewiesen.

§ 6

Unsere Untersuchungen über die minimale Länge der mit $\mathfrak{F}_k$ endenden Q_r-Ketten und Q_l-Ketten der Halbgruppen $\mathfrak{F}_k$ $(k \geqq 1)$ beginnen wir mit einer Modifizierung des Begriffes Q_r-Kette für reguläre Halbgruppen: *Eine Q_r-Kette $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ einer regulären Halbgruppe $\mathfrak{R}$ heißt Q'_r-Kette von $\mathfrak{R}$, wenn sie an Stelle der Forderung* 1) *aus Satz 3 der schärferen Bedingung genügt:*

1′) *Für alle Elemente a und b aus $\mathfrak{R}$ folgt aus $a \cdot b \in \mathfrak{x}_\varkappa$ stets $a \in \mathfrak{x}_\varkappa$ und $b \in \mathfrak{x}_\varkappa$* $(\varkappa = 1, 2, \ldots, k)$.

Hilfssatz 7. *Zu jeder Q_r-Kette $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_k$ einer regulären Halbgruppe $\mathfrak{R}$ gibt es eine Q'_r-Kette $\mathfrak{y}_1 \subseteqq \mathfrak{y}_2 \subseteqq \cdots \subseteqq \mathfrak{y}_k$ von $\mathfrak{R}$ der gleichen Länge k, so daß $\mathfrak{Q}_r^k(\mathfrak{R}; \mathfrak{x}_1, \ldots, \mathfrak{x}_k) = \mathfrak{Q}_r^k(\mathfrak{R}; \mathfrak{y}_1, \ldots, \mathfrak{y}_k)$ gilt.*

Beweis. Wir wählen in jedem Erweiterungsschritt der Quotientenerweiterung von $\mathfrak{R}$ zu der k-ten r-Quotientenhalbgruppe $\mathfrak{Q}_r^k(\mathfrak{R}; \mathfrak{x}_1, \ldots, \mathfrak{x}_k) = \mathfrak{Q}_r^k(\mathfrak{R}; \mathfrak{n}_1, \ldots, \mathfrak{n}_k)$ von $\mathfrak{R}$ an Stelle der jeweiligen Nennermengen $\mathfrak{n}_\varkappa$ die zugehörigen relativ maximalen Nennermengen $\mathfrak{m}_\varkappa$, für die dann ebenfalls $\mathfrak{m}_{\varkappa-1} \subseteqq \mathfrak{m}_\varkappa$ und $\mathfrak{m}_{\varkappa-1}^{-1} \subseteqq \mathfrak{m}_\varkappa$ mit $\varkappa = 2, 3, \ldots, k$ erfüllt ist. Dann bilden die Unterhalbgruppen $\mathfrak{y}_\varkappa = \mathfrak{m}_\varkappa \cap \mathfrak{R}$ eine Q_r-Kette von $\mathfrak{R}$ der Länge k, und es gilt

$$\mathfrak{Q}_r^k(\mathfrak{R}; \mathfrak{n}_1, \ldots, \mathfrak{n}_k) = \mathfrak{Q}_r^k(\mathfrak{R}; \mathfrak{m}_1, \ldots, \mathfrak{m}_k) = \mathfrak{Q}_r^k(\mathfrak{R}; \mathfrak{y}_1, \ldots, \mathfrak{y}_k).$$

Für je zwei Elemente a und b aus $\mathfrak{R}$ folgt aus $a \cdot b \in \mathfrak{y}_\varkappa$ zunächst $a \cdot b \in \mathfrak{m}_\varkappa$, also wegen der relativen Maximalität von $\mathfrak{m}_\varkappa$ gerade $a \in \mathfrak{m}_\varkappa$ und $b \in \mathfrak{m}_\varkappa$, woraus sich sofort $a \in \mathfrak{y}_\varkappa$ und $b \in \mathfrak{y}_\varkappa$ ergibt.

Auf Grund dieses Hilfssatzes und seiner dualen Aussage dürfen wir uns im Folgenden auf die Untersuchung von Q'_r-Ketten und Q'_l-Ketten von $\mathfrak{F}_k$ bzw. $\mathfrak{H}_k$ beschränken. Dabei wird es sich als nützlich erweisen, daß es einen Homomorphismus von $\mathfrak{F}_k$ auf die Unterhalbgruppe $\mathfrak{H}_{k-1} \subseteqq \mathfrak{F}_k$ gibt, der jede Q'_r-Kette und jede Q'_l-Kette von $\mathfrak{F}_k$ in eine ebensolche von $\mathfrak{H}_{k-1}$ gleicher Länge überführt. Die durch

$$\varphi(hC^cD^d) = h \in \mathfrak{H}_{k-1} \quad \text{für alle Elemente} \quad hC^cD^d \in \mathfrak{F}_k$$

definierte eindeutige Abbildung φ von $\mathfrak{F}_k$ auf $\mathfrak{H}_{k-1}$ ist nämlich gemäß

$$\begin{aligned}\varphi(hC^cD^d \cdot h'C^{c'}D^{d'}) &= \varphi(hh'C^{c+c'}D^{d'+2j_{k-1}(h')+c'd}) \\ &= hh' = \varphi(hC^cD^d) \cdot \varphi(h'C^{c'}D^{d'})\end{aligned}$$

ein Homomorphismus von $\mathfrak{F}_k$ auf $\mathfrak{H}_{k-1}$, bei dem sich auch die Forderung 1′) von einer Q'_r-Kette $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_n$ von $\mathfrak{F}_k$ auf ihr homomorphes Bild $\varphi(\mathfrak{x}_1) \subseteqq \cdots \subseteqq \varphi(\mathfrak{x}_n)$ überträgt. Denn gilt $h \cdot h' \in \varphi(\mathfrak{y}_\nu)$ für beliebige Elemente h und h' aus $\mathfrak{H}_{k-1}$ mit $\nu = 1, 2, \ldots, n$, so folgt zunächst, daß $\mathfrak{x}_\nu$ ein Element $hh'C^cD^d$ mit gewissen Exponenten c und d enthält. Dann sind gemäß 1′) auch h und h' Elemente von $\mathfrak{x}_\nu$ und damit auch Elemente von $\varphi(\mathfrak{x}_\nu)$. Weiterhin zieht auch die Gültigkeit der Bedingungen $Q_r^\nu(\mathfrak{F}_k; \mathfrak{x}_1, \ldots, \mathfrak{x}_\nu)$ die Gültigkeit der Bedingungen $Q_r^\nu(\mathfrak{H}_{k-1}; \varphi(\mathfrak{x}_1), \ldots, \varphi(\mathfrak{x}_\nu))$ nach sich; denn sind $a^1 \in \mathfrak{H}_{k-1}$ und $x^1 \in \varphi(\mathfrak{x}_\nu)$ beliebig vorgegebene Elemente, so gibt es wegen $a^1 \in \mathfrak{F}_k$ und $x^1 \in \mathfrak{x}_\nu$ geeignete Elemente aus den entsprechenden Unterhalbgruppen von $\mathfrak{F}_k$, die alle Gleichungen der Bedingung $Q_r^\nu(\mathfrak{F}_k; \mathfrak{x}_1, \ldots, \mathfrak{x}_\nu)$ erfüllen, und damit nach Übergang zu den jeweiligen Bildern bei der homomorphe Abbildung φ von $\mathfrak{F}_k$ auf $\mathfrak{H}_{k-1}$ auch geeignete Elemente aus den entsprechenden Unterhalbgruppen von $\mathfrak{H}_{k-1}$, die alle Gleichungen der Bedingung $Q_r^\nu(\mathfrak{H}_{k-1}; \varphi(\mathfrak{x}_1), \ldots, \varphi(\mathfrak{x}_\nu))$ erfüllen. Die Überlegungen für Q_l'-Ketten verlaufen analog.

Insbesondere folgt aus der Existenz eines solchen Homomorphismus von $\mathfrak{F}_k$ auf $\mathfrak{H}_{k-1}$ und der dualen Aussage über $\mathfrak{H}_k$, daß es für alle $j \leqq k$ möglich ist, die Halbgruppe $\mathfrak{F}_k$ auf ihre Unterhalbgruppe $\mathfrak{F}_j$ bzw. $\mathfrak{H}_j$ homomorph abzubilden, je nachdem ob $k-j$ gerade oder ungerade ist, so daß dabei jede Q_r'-Kette (Q_l'-Kette) von $\mathfrak{F}_k$ in eine Q_r'-Kette (Q_l'-Kette) von $\mathfrak{F}_j$ bzw. $\mathfrak{H}_j$ derselben Länge übergeführt wird.

Ersichtlich hat nun jede Q_r'-Kette von $\mathfrak{F}_1$ mindestens die Länge 1, und für $k \geqq 2$ läßt sich die Frage nach der minimalen Länge einer mit $\mathfrak{F}_k$ endenden Q_r'-Kette von $\mathfrak{F}_k$ jetzt auf die Frage nach der minimalen Länge einer mit $\mathfrak{F}_{k-1}$ endenden Q_l'-Kette von $\mathfrak{F}_{k-1}$ zurückführen, denn jede Q_r'-Kette von $\mathfrak{F}_k$

$$\mathfrak{x}_1 \subseteq \mathfrak{x}_2 \subseteq \ldots \subseteq \mathfrak{x}_p = \mathfrak{F}_k$$

der Länge $p<k$ führt zu einer Q_r'-Kette von $\mathfrak{H}_{k-1}$

$$\varphi(\mathfrak{x}_1) \subseteq \varphi(\mathfrak{x}_2) \subseteq \ldots \subseteq \varphi(\mathfrak{x}_p) = \mathfrak{H}_{k-1}$$

der Länge $p<k$, also zu einer Q_l'-Kette von $\mathfrak{F}_{k-1}$

$$\psi_{k-1}^{-1}(\varphi(\mathfrak{x}_1)) \subseteq \psi_{k-1}^{-1}(\varphi(\mathfrak{x}_2)) \subseteq \ldots \subseteq \psi_{k-1}^{-1}(\varphi(\mathfrak{x}_p)) = \mathfrak{F}_{k-1}$$

der Länge $p < k = (k-1)+1$.

Damit haben wir jetzt nur noch zu zeigen: Für alle $k \geqq 1$ hat jede Q_l'-Kette von $\mathfrak{F}_k$, die mit $\mathfrak{F}_k$ selbst endet, mindestens die Länge $k+1$. Dazu zeigen wir: Für jede Q_l'-Kette $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_p$ von $\mathfrak{F}_k$ der Länge p gilt für alle $s=1, 2, \ldots, p$: $\mathfrak{x}_s$ enthält kein Element B_j und kein Element C_j mit $j \leqq k-s+1$.

In der Tat folgt hieraus, daß C_1 bzw. B_1 nicht in $\mathfrak{x}_s$ mit $s \leqq k$ liegt und damit für die Länge p jeder mit $\mathfrak{x}_p = \mathfrak{F}_k$ endenden Q_l'-Kette von $\mathfrak{F}_k$ gerade $p \geqq k+1$ gilt.

Es sei nun $\mathfrak{x}_1 \subseteqq \cdots \subseteqq \mathfrak{x}_p$ eine Q_l'-Kette von $\mathfrak{F}_k$, für welche die genannte Aussage nicht zutrifft; dabei sei $\mathfrak{x}_s$ diejenige Unterhalbgruppe aus dieser Q_l'-Kette mit dem kleinsten Index s, die wenigstens ein Element C_j oder B_j mit $j \leqq k-s+1$ enthält. Wir skizzieren den weiteren Beweis für den Fall, daß $k-(k-s+1) = s-1$ eine gerade, also s eine ungerade Zahl ist. Wir bilden dann die Halbgruppe $\mathfrak{F}_k$ in der vorn angegebenen Weise homomorph auf ihre Unterhalbgruppe $\mathfrak{F}_{k-s+1}$ ab, wobei die Q_l'-Kette $\mathfrak{x}_1 \subseteqq \mathfrak{x}_2 \subseteqq \cdots \subseteqq \mathfrak{x}_p$ von $\mathfrak{F}_k$ in eine Q_l'-Kette $\mathfrak{x}_1' \subseteqq \mathfrak{x}_2' \subseteqq \cdots \subseteqq \mathfrak{x}_p'$ von $\mathfrak{F}_{k-s+1}$ übergeht und die Unterhalbgruppe $\mathfrak{x}_s'$ ein Element C_j oder B_j mit $j \leqq k-s+1$ enthält. Wir wählen ein solches Element als $x_s^1 \in \mathfrak{x}_s'$ sowie $a^1 = C_{k-s+1} D_{k-s+1} \in \mathfrak{F}_{k-s+1}$ und zeigen, daß die Bedingung $Q_l^s(\mathfrak{F}_{k-s+1}; \mathfrak{x}_1', \ldots, \mathfrak{x}_s')$ nicht erfüllt sein kann. Es gäbe nämlich sonst zu den Elementen $x_s^1 \in \mathfrak{x}_s'$ und $a^1 \in \mathfrak{F}_{k-s+1}$ geeignete Elemente

$$\begin{array}{lll} a^j \in \mathfrak{F}_{k-s+1}, & x_s^j \in \mathfrak{x}_s' & (j = 2, 3, \ldots, s), \\ b^j \in \mathfrak{F}_{k-s+1}, & y_s^j \in \mathfrak{x}_s' & (j = 1, 2, \ldots, s), \\ u_j \text{ und } v_j & \text{aus } \mathfrak{x}_j' & (j = 1, 2, \ldots, s), \end{array}$$

so daß die folgenden Gleichungen erfüllt wären

(4) $$u_\lambda a^\lambda = b^\lambda v_\lambda, \quad u_\lambda x_s^\lambda = y_s^\lambda v_\lambda \qquad (\lambda = 1, 2, \ldots, s),$$

(5) $$u_s = y_s^s, \quad v_s = x_s^s,$$

(6) $$a^{2i} = a^{2i+1}, \quad x_s^{2i} = x_s^{2i+1}, \quad b^{2i-1} = b^{2i}, \quad y_s^{2i-1} = y_s^{2i},$$

wobei in (6) der Index i die Werte $1, 2, \ldots, \frac{s-1}{2}$ durchläuft. Für alle Unterhalbgruppen $\mathfrak{x}_\sigma'$ $(\sigma < s)$ gilt nach Wahl von s, daß sie kein Element C_j und kein Element B_j mit $j \leqq k-s+1 \leqq k-\sigma+1$ enthalten und damit wegen 1′) auch kein Element, in dessen Darstellung irgendeine Potenz mit von 0 verschiedenem Exponenten dieser Elemente vorkommt. Das bedeutet gerade

(7) $$i_{k-s+1}(u_\lambda) = i_{k-s+1}(v_\lambda) = 0 \qquad (\lambda = 1, 2, \ldots, s-1).$$

Hiermit erhalten wir aus den Gleichungen von (4) für $\lambda = 1, 2, \ldots, s-1$

$$i_{k-s+1}(a^\lambda) = i_{k-s+1}(b^\lambda) \quad \text{und} \quad i_{k-s+1}(x_s^\lambda) = i_{k-s+1}(y_s^\lambda).$$

Berücksichtigen wir noch die Gleichungen (6), so erhalten wir

(8) $$\begin{aligned} i_{k-s+1}(a^s) &= i_{k-s+1}(a^{s-1}) = i_{k-s+1}(b^{s-1}) = \ldots = 1 \\ i_{k-s+1}(x_s^s) &= i_{k-s+1}(x_s^{s-1}) = i_{k-s+1}(y_s^{s-1}) = \ldots = 1. \end{aligned}$$

Für jedes Element $f = h C_{k-s+1}^{c_{k-s+1}} D_{k-s+1}^{d_{k-s+1}} \in \mathfrak{F}_{k-s+1}$ (mit $h \in \mathfrak{H}_{k-s}$) setzen wir nun $d(f) = d_{k-s+1}$, und auf Grund der Multiplikationsvorschrift für die Elemente von $\mathfrak{F}_{k-s+1}$ gilt die Regel $d(ff') = d(f') + 2^{i_{k-s+1}(f')} d(f)$. Mit dieser Schreibweise erhalten wir für $\lambda = 1, 2, \ldots, s-1$ aus (4) zusammen mit (7) und (8)

$$d(a^\lambda) \equiv d(v_\lambda) + d(b^\lambda) \quad \text{modulo } 2,$$

$$d(x_s^\lambda) \equiv d(v_\lambda) + d(y_s^\lambda) \quad \text{modulo } 2,$$

also

$$d(a^\lambda) - d(b^\lambda) \equiv d(x_s^\lambda) - d(y_s^\lambda) \quad \text{modulo } 2.$$

Mit Hilfe von (6) erhält man schließlich

$$d(a^1) - d(a^{s-1}) \equiv d(x_s^1) - d(x_s^{s-1}) \quad \text{modulo } 2. \tag{9}$$

Wir ziehen nun noch die erste der beiden noch nicht betrachteten Gleichungen für $\lambda = s$ aus (4) heran, die wegen (5) die Form $y_s^s a^s = b^s x_s^s$ annimmt und

$$d(a^s) + 2^{i_{k-s+1}(a^s)} d(y_s^s) = d(x_s^s) + 2^{i_{k-s+1}(x_s^s)} d(b^s),$$

also wegen (8)

$$d(a^s) \equiv d(x_s^s) \quad \text{modulo } 2 \tag{10}$$

liefert. Aus (9) und (10) erhalten wir schließlich wegen $a^s = a^{s-1}$ und $x_s^s = x_s^{s-1}$ (gemäß (6))

$$d(a^1) \equiv d(x_s^1) \quad \text{modulo } 2,$$

während doch einerseits $d(a^1) = d(C_{k-s+1} D_{k-s+1}) = 1$ und andererseits $d(x_s^1) = 0$ wegen $x_s^1 = C_j$ oder $x_s^1 = B_j$ gilt. In ähnlicher Weise führt man auch den Fall zum Widerspruch, daß $k - (k - s + 1) = s - 1$ ein ungerade Zahl, also s eine gerade Zahl ist, nur bildet man jetzt die Halbgruppe $\mathfrak{F}_k$ in der vorn angegebenen Weise homomorph auf ihre Unterhalbgruppe $\mathfrak{H}_{k-s+1}$ ab. Man wählt dann $a^1 = A_{k-s+1} B_{k-s+1} \in \in \mathfrak{H}_{k-s+1}$ und zeigt, daß die Bedingung $Q_l^s(\mathfrak{H}_{k-s+1}; x_1', \ldots, x_s')$ nicht erfüllt sein kann.

Die in II) ausgesprochenen Behauptungen über die Halbgruppen $\mathfrak{F}_k$ sind damit bewiesen, womit wegen der Dualität der Halbgruppen $\mathfrak{F}_k$ und $\mathfrak{H}_k$ auch die Aussagen von II) über die Halbgruppen $\mathfrak{H}_k$ richtig sind.

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On Σ-ordered inverse semigroups

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In this paper we define Σ-ordered inverse semigroups (Definition 1) which are partially ordered inverse semigroups whose partial orders are completely determined by families of normal sub-semigroups (Theorem 2). The set of normal sub-semigroups determining the partial order is analogous to the positive cone in a partially ordered group ([2]). We next consider the set $\mathfrak{O}(X)$ of all partial o-isomorphism between o-subsets (Definition 6) of a partially ordered set $(X, \leqq)$. This set is an inverse sub-semigroup of the symmetric inverse semigroup $\mathfrak{I}(X)$ on X. A partial order is defined on $\mathfrak{O}(X)$ in a natural way which makes $\mathfrak{O}(X)$ a Σ-ordered inverse semigroup (Theorem 11). We call $\mathfrak{O}(X)$ the symmetric Σ-ordered inverse semigroup on $(X, \leqq)$. Finally, we prove the Preston—Vagner theorem for Σ-ordered inverse semigroups claiming that any Σ-ordered inverse semigroup can be embedded o-isomorphically into a symmetric Σ-ordered inverse semigroup of partial o-isomorphisms (Theorem 12). Questions of order theoretic interest will be studied separately and are not dealt with in this paper.

For terminology and information on semigroups we refer to [1].

Let S be an inverse semigroup and E the set of idempotents of S. Let Σ be the lattice of all idempotent separating congruence on S with greatest element μ. If M_e $(e \in E)$ is the μ-class containing e, it is known that M_e is a normal subgroup of H_e.

Definition 1. A partially ordered inverse semigroup $(S, \leqq)$ is called a *Σ-ordered inverse semigroup* (and $\leqq$ a Σ-order on S) if the following conditions hold in S:

(1) $a \leqq b \Rightarrow a \mu b$.

(2) If $Q_e = \{x \in S \mid e \leqq x\}$, then $a^{-1} Q_e a \subseteqq Q_{a^{-1}ea}$ for all $e \in E$ and $a \in S$.

Note that in a Σ-ordered semigroup no two idempotents are comparable.

Theorem 2. *Let S be an inverse semigroup and $\{Q_e : e \in E\}$ a collection of subsets which satisfies the following conditions:*

(i) *Q_e is a sub-semigroup of M_e containing e and $Q_e \cdot Q_f \subseteqq Q_{ef}$.*

(ii) *$a^{-1} Q_e a \subseteqq Q_{a^{-1}ea}$ for all $e \in E$ and $a \in S$.*

(iii) *$Q_e \cap Q_e^{-1} = \{e\}$, where Q_e^{-1} is the set of inverses of elements of Q_e in H_e. Then there exists a partial order $\leqq$ on S such that $Q_e = \{x \in S \mid e \leqq x\}$ and $(S, \leqq)$ is a Σ-ordered inverse semigroup.*

Proof. If a partial order $\leqq$ exists on S such that $(S, \leqq)$ is a partially ordered inverse semigroup, then it is clear that $(S, \leqq)$ is a Σ-ordered inverse semigroups. Hence it is enough to establish the existence of such a partial order on S. Let $a, b \in S$. Say $a \leqq b \Leftrightarrow a\mu b,\ ba^{-1} \in Q_{aa^{-1}},\ a^{-1}b \in Q_{a^{-1}a}$. Clearly $a \leqq a$. Let $a, b \in S$, $a \leqq b$ and $b \leqq a$. Then $a \leqq b \Rightarrow a\mu b,\ ba^{-1} \in Q_{aa^{-1}},\ a^{-1}b \in Q_{a^{-1}a}$; $b \leqq a \Rightarrow b\mu a,\ ab^{-1} \in \in Q_{bb^{-1}},\ b^{-1}a \in Q_{b^{-1}b}$. Now, $a\mu b \Rightarrow aa^{-1} = bb^{-1} = e$ and $a^{-1}a = b^{-1}b = f$. ba^{-1} and ab^{-1} are inverses of each other in Q_e and $a^{-1}b$ and $b^{-1}a$ are inverses of each other in Q_f and so $a = b$. Thus $\leqq$ is asymmetric. To show transitivity, let $a \leqq b$, $b \leqq c$,

$$a \leqq b \Rightarrow a\mu b, \qquad ba^{-1} \in Q_{aa^{-1}}, \quad a^{-1}b \in Q_{a^{-1}a}.$$
$$b \leqq c \Rightarrow b\mu c, \qquad cb^{-1} \in Q_{bb^{-1}}, \quad b^{-1}c \in Q_{b^{-1}b}$$

Thus $a\mu b\mu c$ and $aa^{-1} = bb^{-1} = cc^{-1} = e$ and $a^{-1}a = b^{-1}b = c^{-1}c = f$ so, $ca^{-1} = = cb^{-1}ba^{-1} \in Q_e \cdot Q_e \subseteqq Q_e$ and $a^{-1}c = a^{-1}bb^{-1}c \in Q_f \cdot Q_f \subset Q_f$ and so $a \leqq c$. Thus $\leqq$ is a partial order on S.

If $x \in S$ and $e \leqq x$, then $x\mu e$ and $x = xe = ex \in Q_e$ and so $\{x \in S \mid e \leqq x\} \subseteqq Q_e$. On the otherhand if $x \in Q_e$, then $x\mu e$ and $xe = ex = x \in Q_e$ and so $e \leqq x$ and $Q_e \subseteqq \subseteqq \{x \in S \mid e \leqq x\}$. Thus we have $Q_e = \{x \in S \mid e \leqq x\}$.

Let $a \leqq b$, $c \in S$. Then $a \leqq b \Rightarrow a\mu b,\ ba^{-1} \in Q_{aa^{-1}},\ a^{-1}b \in Q_{a^{-1}a}$; also $a\mu b \Rightarrow ac\mu bc$: $(bc)(ac)^{-1} = bcc^{-1}a^{-1} = ba^{-1}acc^{-1}a^{-1} \in Q_{aa^{-1}} \cdot Q_{(ac)(ac)^{-1}} \subseteqq Q_{(ac)(ac)^{-1}}$; $(ac)^{-1}(bc) = c^{-1}a^{-1}bc \in c^{-1}Q_{a^{-1}a} \cdot c \subseteqq Q_{(ac)^{-1}(ac)}$. Thus $ac \leqq bc$. Similarly we can show that $ca \leqq cb$. Hence $(S, \leqq)$ is a partially ordered inverse semigroup. It now follows that $(S, \leqq)$ is a Σ-ordered inverse semigroup.

Let $(S, \leqq)$ be a Σ-ordered inverse semigroup. Then,

Lemma 3. $a \leqq b \Rightarrow b^{-1} \leqq a^{-1}$.

Proof. $a \leqq b \Rightarrow a\mu b \Rightarrow a^{-1}\mu|b^{-1}|$. Hence $b^{-1} = b^{-1}aa^{-1} \leqq b^{-1}ba^{-1} = a^{-1}$.

Proposition 4. *If T is an inverse sub-semigroup of $(S, \leqq)$, then $(T, \leqq)$ is a Σ-ordered inverse semigroup.*

Proof. Let μ_0 be the maximum idempotent separating congruence on T and μ_T the restriction of μ to T. Then $\mu_T \subseteqq \mu_0$. Let $Q_e^T = Q_e \cap T$. Now $Q_e^T \neq \emptyset \Leftrightarrow e \in T$. The nonempty sets Q_e^T $(e \in E \cap T)$ satisfy the conditions of Theorem 2 and so there exists a partial order $\leqq'$ on T such that $(T, \leqq')$ is a Σ-ordered inverse semigroup. We now show that $\leqq$ coincides with $\leqq'$ on T. It is immediate that $a \leqq b$ $(a, b \in T) \Rightarrow \Rightarrow a \leqq' b$. Conversely if $a, b \in T$ then

$$a \leqq' b \Rightarrow \{a\mu_0 b,\ ba^{-1} \in Q^T_{aa^{-1}},\ a^{-1}b \in Q^T_{a^{-1}a}\} \Rightarrow$$
$$\Rightarrow \{aa^{-1} = bb^{-1} = e,\ a^{-1}a = b^{-1}b = f,\ ba^{-1} \in Q_e \subset M_e,\ a^{-1}b \in Q_f \subseteqq M_f\} \Rightarrow$$
$$\Rightarrow \{a\mu b,\ a^{-1}b \in Q_f,\ ba^{-1} \in Q_e\} \Rightarrow a \leqq b.$$

The following lemma follows immediately.

Lemma 5. *$(H_e, \leqq)$, where H_e is the maximal subgroup containing e in $(S, \leqq)$, is a partially ordered group. Further, if $e\mathcal{D}f$, then H_e and H_f are o-isomorphic.*

We shall now describe the symmetric Σ-ordered inverse semigroup on a partially ordered set. Let $(X, \leqq)$ be a partially ordered set. Let $\mathfrak{I}(X)$ denote the symmetric inverse semigroup of all partial $(1-1)$ transformations on X. If $\alpha \in \mathfrak{I}(X)$ denote by $\Delta(\alpha)$ and $\nabla(\alpha)$ the domain and the range of α, respectively.

Definition 6. A subset $A \subseteqq (X, \leqq)$ is called an *o-subset* of X if $a \in A$, $a \leqq b$ (or $b \leqq a$) implies $b \in A$.

It is clear that the set of all o-subsets of $(X, \leqq)$ is closed for the operations of set-intersection and set union and contains the null set φ and X.

Let $\mathfrak{O}(X)$ denote the subset of $\mathfrak{I}(X)$ consisting of all o-isomorphisms between o-subsets of X. $\mathfrak{O}(X)$ contains the map 0 and the identity map of X.

Proposition 7. *$\mathfrak{O}(X)$ is an inverse sub-semigroup of $\mathfrak{I}(X)$.*

Proof. If $\alpha \in \mathfrak{O}(X)$, then α^{-1} is also an o-isomorphism between o-subsets and so belongs to $\mathfrak{O}(X)$. Thus it is enough to show that $\mathfrak{O}(X)$ is a subsemigroup of $\mathfrak{I}(X)$. Let $\alpha, \beta \in \mathfrak{O}(X)$ and $A = \nabla(\alpha) \cap \Delta(\beta)$. If $A = \emptyset$, then $\alpha\beta = 0 \in \mathfrak{O}(X)$. If $A \neq \emptyset$, let $A_1 = A\alpha^{-1}$, $A_2 = A\beta$. A is an o-subset of X. A_1, and A_2 are also o-subsets. For, let $x \in A_1$, $y \leqq x$. Since $A_1 \subset \Delta(\alpha)$, $y \in \Delta(\alpha)$ and so $y\alpha \leqq x\alpha \in A$. Since A is an o-subset, $y\alpha \in A$ and so $y \in A\alpha^{-1} = A_1$. Now $A_1 = \Delta(\alpha\beta)$, $A_2 = \nabla(\alpha\beta)$ and $\alpha\beta : A_1 \to A_2$ is clearly an o-isomorphism with $\beta^{-1}\alpha^{-1}$ as its inverse. Thus $\mathfrak{O}(X)$ is an inverse sub-semigroup of $\mathfrak{I}(X)$.

Definition 8. For $\alpha, \beta \in \mathfrak{O}(X)$ put $\alpha \leqq \beta \Leftrightarrow \alpha\mathscr{H}\beta$, $x\alpha \leqq x\beta$ for all $x \in \Delta(\alpha)$ and $y\beta^{-1} \leqq y\alpha^{-1}$ for all $y \in \nabla(\alpha)$.

Note that $\alpha\mathscr{H}\beta \Leftrightarrow \Delta(\alpha) = \Delta(\beta)$ and $\nabla(\alpha) = \nabla(\beta)$.

Proposition 9. *$\alpha \leqq \beta \Rightarrow \alpha\mu\beta$, where μ is the maximum idempotent separating congruence on $\mathfrak{O}(X)$.*

Proof. We will show that $\alpha \leqq \beta$ implies $\alpha^{-1}\varepsilon\alpha = \beta^{-1}\varepsilon\beta$ for all idempotents ε in $\mathfrak{O}(X)$ which proves that $\alpha\mu\beta$ ([1] Lemma 7. 57). $\alpha \leqq \beta \Rightarrow \alpha\mathscr{H}\beta$, $x\alpha \leqq x\beta$ for all $x \in \Delta(\alpha)$ and $y\beta^{-1} \leqq y\alpha^{-1}$ for all $y \in \nabla(\alpha)$. Let $A : \Delta(\varepsilon) \cap \Delta(\alpha) = \Delta(\varepsilon) \cap \Delta(\beta)$ and $B = A\varepsilon \cap \Delta(\alpha) = A \cap \Delta(\alpha)$, since ε is identity on A. Then ε is also identity on B and so $\alpha^{-1}\varepsilon\alpha$ is the identity map of $B\alpha$ and $\beta^{-1}\varepsilon\beta$ is the identity map of $B\beta$. Since $\alpha, \beta, \varepsilon$ are o-isomorphism and $\alpha \leqq \beta$ we have A and B are o-subsets and so $B\alpha = B\beta$. Hence $\alpha^{-1}\varepsilon\alpha = \beta^{-1}\varepsilon\beta$.

Proposition 10. *Let $Q_\varepsilon = \{\alpha \in \mathfrak{O}(X) | \varepsilon \leqq \alpha\}$, where ε is an idempotent of $\mathfrak{O}(X)$. Then $Q_\varepsilon \cdot Q_{\varepsilon_1} \subseteqq Q_{\varepsilon\varepsilon_1}$, $v^{-1} Q_\varepsilon v \subseteqq Q_{v^{-1}\varepsilon v}$ for all elements v and idempotents $\varepsilon, \varepsilon_1 \in \mathfrak{O}(X)$.*

Proof. Let $\alpha \in Q_\varepsilon$, $\beta \in Q_{\varepsilon_1}$. Then $\varepsilon\mu\alpha$ and $\varepsilon_1\mu\beta$ and so $\varepsilon\varepsilon_1\mu\alpha\beta$. Now $\Delta(\varepsilon) \cap \Delta(\varepsilon_1) = \Delta(\varepsilon\varepsilon_1) = \Delta(\alpha\beta) = \nabla(\alpha\beta)$. $x \in \Delta(\varepsilon\varepsilon_1) \Rightarrow x \leqq x\alpha$ and $x \leqq x\beta \Rightarrow x \leqq x\beta \leqq x\alpha\beta$. $y \in \Delta(\varepsilon\varepsilon_1) \Rightarrow y\alpha^{-1} \leqq y$ and $y\beta^{-1} \leqq y \Rightarrow y\beta^{-1}\alpha^{-1} \leqq y$. Thus $\varepsilon\varepsilon_1\mu\alpha\beta$, $x\varepsilon\varepsilon_1 \leqq x\alpha\beta$ and $x\beta^{-1}\alpha^{-1} \leqq x\varepsilon\varepsilon_1$ for all $x \in \Delta(\varepsilon\varepsilon_1)$ and hence $\varepsilon\varepsilon_1 \leqq \alpha\beta$.

Let $\alpha \in Q_\varepsilon$ and $\nu \in \mathfrak{D}(X)$. Then $\nu^{-1}\varepsilon\nu\mu\nu^{-1}\alpha\nu$. So, $\Delta(\nu^{-1}\varepsilon\nu)=\Delta(\nu^{-1}\alpha\nu)=\nabla(\nu^{-1}\alpha\nu)$. $x\in\Delta(\nu^{-1}\varepsilon\nu)$ and $\varepsilon\leqq\alpha \Rightarrow x\nu^{-1}\in\Delta(\varepsilon)\cap\Delta(\nu)=\Delta(\alpha)\cap\Delta(\nu)\Rightarrow x\nu^{-1}\varepsilon\leqq x\nu^{-1}\alpha$ in $\Delta(\nu)\Rightarrow x\nu^{-1}\varepsilon\nu\leqq x\nu^{-1}\alpha\nu$. $y\in\Delta(\nu^{-1}\varepsilon\nu)\Rightarrow y\nu^{-1}\alpha^{-1}\leqq y\nu^{-1}\varepsilon\Rightarrow y\nu^{-1}\alpha^{-1}\nu\leqq y\nu^{-1}\varepsilon\nu$. Thus $\nu^{-1}\varepsilon\nu\leqq\nu^{-1}\alpha\nu$.

Theorem 11. $(\mathfrak{D}(X), \leqq)$ *is a Σ-ordered inverse semigroup. It is called the symmetric Σ-ordered inverse semigroup on* $(X, \leqq)$.

Proof. From propositions 9 and 10 it follows that the sets $Q_\varepsilon=\{\alpha\in\mathfrak{D}(X)|\varepsilon\leqq\alpha\}$ satisfy the conditions (i) and (ii) of Theorem 2. To show condition (iii) consider $\alpha, \alpha^{-1}\in Q_\varepsilon$. $x\in\Delta(\varepsilon)$, $\varepsilon\leqq\alpha\Rightarrow x=x\varepsilon\leqq x\alpha\Rightarrow x\varepsilon\alpha^{-1}\leqq x\alpha\alpha^{-1}=x\varepsilon$; and $\varepsilon\leqq\alpha^{-1}\Rightarrow x= =x\varepsilon\leqq x\alpha^{-1}\Rightarrow x\varepsilon\alpha\leqq x\alpha^{-1}\alpha=x\varepsilon$. Thus we have $x\varepsilon\leqq x\alpha\leqq x\varepsilon$ and $x\varepsilon\leqq x\alpha^{-1}\leqq x\varepsilon$ for all $x\in\Delta(\varepsilon)$ and so $\alpha=\varepsilon=\alpha^{-1}$. It then follows that $\leqq$ is a Σ-order on $\mathfrak{D}(X)$ defined by the sets $\{Q_\varepsilon\}$. Hence $(\mathfrak{D}(X), \leqq)$ is a Σ-ordered inverse semigroup.

We now consider representation of Σ-ordered inverse semigroups by partial transformation.

Proposition 12. *If $(S, \leqq)$ is a Σ-ordered inverse semigroup, then the mapping $\varrho_a: x\to xa$ of Sa^{-1} onto Sa, belongs to $\mathfrak{D}(S)$ for every $a\in S$.*

Proof. The sets Sa $(a\in S)$ are o-subsets of $(S, \leqq)$. For, $x\in Sa$, $y\leqq x$ (or $x\leqq y$) $\Rightarrow$ $\Rightarrow x\mu y\Rightarrow Sy=Sx\subseteqq Sa\Rightarrow y\in Sa$. Further ϱ_a and $\varrho_{a^{-1}}$ are order preserving maps which are inverse of each other. Hence $\varrho_a\in\mathfrak{D}(S)$:

Theorem 12. *A Σ-ordered inverse semigroup $(S, \leqq)$ is o-isomorphic to a Σ-ordered inverse sub-semigroup of the symmetric Σ-ordered inverse semigroup $(\mathfrak{D}(S), \leqq)$.*

Proof. The mapping $\varrho: a\to\varrho_a$ of S into $\mathfrak{D}(S)$ is clearly an isomorphism of the inverse semigroups. We now show that for $a, b\in S$, $a\leqq b\Leftrightarrow\varrho_a\leqq\varrho_b\cdot a\leqq b\Rightarrow$ $\Rightarrow a\mu b\Rightarrow Sa=Sb$, $Sa^{-1}=Sb^{-1}\Rightarrow\varrho_a\mathscr{H}\varrho_b$. Further, if $x\in Sa^{-1}$, then $x\varrho_a=xa\leqq$ $\leqq xb=x\varrho_b$ and if $y\in Sa$, $y\varrho_b^{-1}=y\varrho_{b^{-1}}=yb^{-1}\leqq ya^{-1}\Rightarrow y\varrho_{a^{-1}}=y\varrho_a^{-1}$. Hence $\varrho_a\leqq\varrho_b$. Next, if $\varrho_a\leqq\varrho_b$, then $Sa^{-1}=\Delta(\varrho_a)=\Delta(\varrho_b)=Sb^{-1}$ and $Sa=\nabla(\varrho_a)=\nabla(\varrho_b)=Sb$, and for all $x\in Sa$, $xa\leqq xb$. Hence $aa^{-1}=bb^{-1}=e$ and $a^{-1}a=b^{-1}b=f$ and so $ea\leqq$ $\leqq eb\cdot i\cdot ea\leqq b$. Hence ϱ is an o-isomorphism of $(S, \leqq)$ into $(\mathfrak{D}(S), \leqq)$.

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Thin operators in a von Neumann algebra

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1. Introduction. Let $\mathscr{A}$ be a von Neumann algebra, $\mathscr{I}$ a uniformly closed two-sided ideal in $\mathscr{A}$ and $\mathscr{P}$ the lattice of projections in $\mathscr{I}$. Let the center, $\mathscr{Z}$, of $\mathscr{A}$ be identified with $C(\Omega)$, the algebra of all continuous complex-valued functions on some Hyperstonian space [3]. We say that $A \in \mathscr{A}$ is *thin* relative to $\mathscr{I}$ if $A = Z+K$, $Z \in \mathscr{Z}$, $K \in \mathscr{I}$. It is shown that the thin operators relative to $\mathscr{I}$ form a C^*-subalgebra of $\mathscr{A}$. The lattice $\mathscr{P}$ is a directed set under the usual ordering (if $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$, the algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$, then $P \leqq Q$ means $(Px, x) \leqq (Qx, x)$ all $x \in \mathscr{H}$). It was conjectured by P. R. HALMOS, and proved by R. G. DOUGLAS and C. PEARCY [5] for $\mathscr{A} = \mathscr{B}(\mathscr{H})$, $\mathscr{H}$ separable, and $\mathscr{I}$ the ideal of compact operators, that A is thin relative to $\mathscr{I}$ if and only if

$$(H) \qquad \lim_{P \in \mathscr{P}} \|PAP - AP\| = 0.$$

Douglas and Pearcy asked whether (H) characterizes the thin operators relative to an arbitrary uniformly closed ideal in any von Neumann algebra. It is the purpose of this note to show that this characterization holds for any such ideal in a von Neumann factor. Also, it is proved for maximal ideals in certain more general von Neumann algebras, and for certain ideals in type I algebras.

2. Let $\mathscr{I}$ be a uniformly closed ideal in a von Neumann algebra $\mathscr{A}$. The set of thin operators $\mathscr{I} + \mathscr{Z}$ forms a C^*-subalgebra of $\mathscr{A}$ [4, 1. 8. 4]. There is a two-sided (not necessarily closed) ideal $\mathscr{L}$ of $\mathscr{A}$ with the property that $T \in \mathscr{L}$ if and only if the range projection of T is also in $\mathscr{L}$; furthermore, $\mathscr{I}$ is the uniform closure of $\mathscr{L}$. This fact, due to W. WILS [10, p. 56, Theorem 1. 4] will be used in the proof of the following proposition.

Proposition 2. 1. *Let $\mathscr{A}$ be any von Neumann algebra, $\mathscr{I}$ any uniformly closed ideal in $\mathscr{A}$. If A is thin relative to $\mathscr{I}$, then A satisfies (H).*

Proof. Let $A = Z+K$, $Z \in \mathscr{Z}$, $K \in \mathscr{I}$. Note that $\|PAP - AP\| = \|(I-P)AP\| = \|(I-P)KP\|$, so it is enough to show $\lim_{P \in \mathscr{P}} \|(I-P)KP\| = 0$. Let $\varepsilon > 0$ be given. It suffices to find a $P_0 \in \mathscr{P}$ such that $Q \in \mathscr{P}$ and $Q > P_0$ implies $\|(I-Q)KQ\| < \varepsilon$. Then for any $P \in \mathscr{P}$, $P_0 \vee P \in \mathscr{P}$, $P_0 \vee P > P$, and if $Q \in \mathscr{P}$ with $Q > P_0 \vee P$, then $\|(I-Q)KQ\| < \varepsilon$.

By the theorem of Wils, choose $T \in \mathcal{I}$ with $P_0 = \mathrm{rp}(T) \in \mathcal{I}$, and $\|T-K\| < \varepsilon$. Then

$$\|(I-P_0)K\| = \|(I-P_0)(T-K)\| < \varepsilon.$$

Now, if $Q \in \mathcal{P}$ and $Q > P_0$, then $I-Q \leqq I-P_0$, so

$$\|(I-Q)KQ\| \leqq \|(I-Q)K\| \leqq \|(I-P_0)K\| < \varepsilon.$$

Hence the proposition follows.

It is easy to see that the converse of Proposition 2. 1 is usually false if $\mathcal{I}$ is not weakly dense in $\mathcal{A}$. In this case, each $A \in \mathcal{A}$ with $A\mathcal{I} = \{0\} = \mathcal{I}A$ satisfies (H).

The techniques in the proof of the next proposition are adapted from those in [1], [5], and [6]. As in [1], we define for any $B \in \mathcal{B}(\mathcal{H})$, any projection $P \in \mathcal{B}(\mathcal{H})$.

$$\eta_B(P\mathcal{H}) = \sup_{x \in P\mathcal{H}, \|x\|=1} \|Bx-(Bx, x)x\|.$$

Proposition 2. 2. *Let $\mathcal{A}$ be a von Neumann algebra, $\mathcal{I}$ a uniformly closed ideal in $\mathcal{A}$, and φ an irreducible representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with $\varphi(\mathcal{I}) \neq 0$. Then for any $A \in \mathcal{A}$,*

$$\inf_{P \in \mathcal{P}} \eta_{\varphi(A)}(\varphi(1-P)\mathcal{H}) \leqq \limsup_{P \in \mathcal{P}} \|PA(I-P)\|.$$

Proof. Let $\varepsilon > 0$ and $P_0 \in \mathcal{P}$ be given. It suffices to show that for each unit vector x in $\varphi(1-P_0)\mathcal{H}$, there is a projection $Q \in \mathcal{P}$, $Q > P_0$, with

$$\|\varphi(A)x-(\varphi(A)x, x)x\| \leqq \|QA(I-Q)\| + \varepsilon.$$

For, we then have that

$$\eta_{\varphi(A)}(\varphi(I-P_0)\mathcal{H}) \leqq \sup_{Q > P_0} \|QA(I-Q)\| + \varepsilon,$$

and in particular, there is some $Q_0 > P_0$ in $\mathcal{P}$ with

$$\eta_{\varphi(A)}(\varphi(I-P_0)\mathcal{H}) \leqq \|Q_0A(I-Q_0)\| + 2\varepsilon.$$

From this it follows that

$$\inf_{P \in \mathcal{P}} \eta_{\varphi(A)}(\varphi(I-P)\mathcal{H}) \leqq \limsup_{P \in \mathcal{P}} \|PA(I-P)\|.$$

Fix $P \in \mathcal{P}$ and $\varepsilon > 0$. Let x be an arbitrary fixed unit vector in $\varphi(I-P)\mathcal{H}$. (If $\varphi(I-P)\mathcal{H} = 0$, we are done.) Set $y = \varphi(A)x-(\varphi(A)x, x)x$. By a theorem of Kadison [9, p. 274, Theorem 1] there is a self-adjoint operator $C \in \mathcal{I}$ such that $\varphi(C)$ is equal to the one-dimensional projection with range spanned by y at the two points x and y; that is, $\varphi(C)y = y$ and $\varphi(C)x = 0$. By considering C^2, we may assume C is a positive operator.

Following the argument in [6, p. 61, Proposition 3. 1], we may assume $0 \leqq C \leqq 1$. For, let f be the continuous real-valued function defined on the interval $[0, \|C\|+1]$

by $f\equiv 0$ on $\left[0, \frac{1}{2}\right]$, $f\equiv 1$ on $\left[\frac{3}{4}, \|C\|+1\right]$ and f linear between $\frac{1}{2}$ and $\frac{3}{4}$. Then f is a uniform limit of polynomials $\{p_n\}$ with real coefficients and with no constant terms. Thus if $p_n(t)=\Sigma a_m t^m$, then

$$\varphi(p_n(C))y = \Sigma a_m \varphi(C)^m y = p_n(I)y \quad \text{and} \quad \varphi(p_n(C)x)=0.$$

Since $p_n(I)=1$, we have

$$\varphi(f(C))y=\lim \varphi(p_n(C))y=y \quad \text{and} \quad \varphi(f(C))x=0.$$

Let $E(\lambda)$ be the spectral resolution of C, and set $E=E((\delta, 1])$. Then $E\in\mathcal{I}$ [2, p. 855, Lemma 4. 1], and $\delta E \leqq C \leqq E+\delta$. Thus

$$\|y\|^2 = \|\varphi(C)y\|^2 = (\varphi(C)y, y) \leqq ((\varphi(E)+\delta)y, y) = \|\varphi(E)y\|^2+\delta\|y\|^2.$$

Thus for sufficiently small $\delta>0$, we have $\|\varphi(E)y\| \geqq \|y\|-\varepsilon$. Furthermore,

$$\delta^2\|\varphi(E)x\|=(\varphi(\delta E)x, x)\leqq(\delta(C)x, x)=0.$$

Now set $Q = E\vee P$. Then $Q\in\mathcal{I}$,

$$\|\varphi(Q)y\| \geqq \|\varphi(E)y\| \geqq \|y\|-\varepsilon \quad \text{and} \quad \varphi(Q)x=0.$$

Thus,

$$\begin{aligned}\|QA(I-Q)\| &\geqq \|\varphi(QA(I-Q))x\| = \|\varphi(QA)x\|\\ &= \|\varphi(Q)(\varphi(A)x-(\varphi(A)x, x)x)+\varphi(Q)(\varphi(A)x, x)x\| = \|\varphi(Q)y\| \geqq \|y\|-\varepsilon.\end{aligned}$$

So, $\|QA(I-Q)\|+\varepsilon \geqq \|\varphi(A)x-(\varphi(A)x, x)x\|$, and the proof is complete.

Note that the following theorem does not conflict with Proposition 2. 1, when $\mathcal{A}$ has non-trivial center $\mathcal{Z}=C(\Omega)$. The hypothesis that $\mathcal{I}$ contain a primitive ideal insures that $\mathcal{I}\cap\mathcal{Z}$ is some maximal ideal $\mu\in\Omega$. Hence for any $Z\in\mathcal{Z}$, we have $Z(\mu)\in\mathbf{C}$, and $Z-Z(\mu)\in\mathcal{I}$.

Theorem 2. 3. *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{I}$ a uniformly closed ideal in $\mathcal{A}$ which properly contains a primitive ideal of $\mathcal{A}$. If $A\in\mathcal{A}$ satisfies (H), then $A = \lambda+K$, $K\in\mathcal{I}$, λ a scalar operator.*

Proof. Note that $\|PA^*(I-P)\| = \|PAP-AP\|$ and that $A^* = \lambda+K$ implies $A = \bar{\lambda}+K^*$. Hence it is equivalent to assume that $\lim_{P\in\mathcal{P}} \|PA(I-P)\| = 0$, and show that $A = \lambda+K$, λ a scalar, $K\in\mathcal{I}$.

Let φ be the irreducible representation of $\mathcal{A}$ on $\mathcal{H}$ whose kernel is the primitive ideal properly contained in $\mathcal{I}$. Thus $\varphi(\mathcal{I})\neq\{0\}$, and by Proposition 2. 2,

$$\inf_{P\in\mathcal{P}} \eta_{\varphi(A)}(\varphi(I-P)\mathcal{H}) = 0.$$

In a particular, there is a sequence of projections $\{P_n\} \subset \mathscr{P}$ with

$$\eta_n = \eta_{\varphi(A)}(\varphi(I-P_n)\mathscr{H}) < 1/n, \quad n=1, 2, \ldots;$$

and we may assume the sequence $\{P_n\}$ is increasing.

Following the proof of [1, p. 115 Theorem 1] we have that

$$W_{\varphi(A)}(\varphi(I-P_n)\mathscr{H}) = \{(\varphi(A)x, x): x \in \varphi(I-P_n)\mathscr{H}, \|x\| = 1\}$$

is a nested sequence of convex sets. Furthermore, by [1, p. 114, Lemma 2. 2],

$$\text{diameter } W_{\varphi(A)}(\varphi(I-P_n)\mathscr{H}) \leqq 8\|\varphi(A)\|^{\frac{1}{2}}\eta_n.$$

Hence there is a unique complex number λ which is adherent to every $W_{\varphi(A)}(\varphi(I-P_n)\mathscr{H})$. Set $K = A-\lambda$, $\varphi(K) = \varphi(A)-\lambda$. Applying [1, p. 115, Lemma 2. 3], we have

$$\|\varphi(K)(I-\varphi(P_n))\|^2 \leqq 65\|\varphi(A)\|\eta_n.$$

Thus $\{\varphi(K)(I-\varphi(P_n))\}$ converges to zero. Since $\varphi(KP_n) \in \varphi(\mathscr{I})$ for each n, where $\varphi(\mathscr{I})$ is uniformly closed, we see that $\varphi(K) \in \varphi(\mathscr{I})$. But ker $(\varphi) \subset \mathscr{I}$ then implies that $K \in \mathscr{I}$. Since $A = K+\lambda$, the proof is complete.

Theorem 2. 4. *Let $\mathscr{A}$ be a von Neumann factor and $\mathscr{I}$ any uniformly closed ideal in $\mathscr{A}$. Then $A \in \mathscr{A}$ satisfies (H) if and only if $A = \lambda + K$, $K \in \mathscr{I}$, λ a scalar operator.*

Proof. Let Ψ be a non-zero irreducible representation of the C^*-algebra $\mathscr{I}$ on some Hilbert space $\mathscr{H}$. Let φ be an extension of Ψ to an irreducible representation of $\mathscr{A}$ on $\mathscr{H}$ [4, 2. 10. 2 and 2. 11. 3]. The uniformly closed ideals in the factor $\mathscr{A}$ are totally ordered [12]. Since the kernel of φ is such an ideal, $\mathscr{I} \nsubseteq \ker \varphi$ implies $\ker \varphi \subset \mathscr{I}$. Thus the theorem follows by Theorem 2. 3 and Proposition 2. 1.

Theorem 2. 5. *Let $\mathscr{A}$ be a semifinite, properly infinite von Neumann algebra, or a type III algebra with no σ-finite central projection. If $\mathscr{M}$ is a maximal ideal in $\mathscr{A}$, then $A \in \mathscr{A}$ is thin relative to $\mathscr{M}$ if and only if $A = \lambda + M$, λ a scalar and $M \in \mathscr{M}$.*

Proof. We may assume A is not a factor, by the preceding theorem. Let $\mathscr{J}$ denote the strong radical of $\mathscr{A}$; then $\mathscr{J} \neq \{0\}$. In fact, if $\mathscr{A}$ is semi-finite, $\mathscr{J}$ contains all the finite projections of $\mathscr{A}$ [6, p. 55—56 and p. 58, Proposition 2. 3]. Thus every projection in $\mathscr{A}$ dominates a projection in $\mathscr{J}$. This is also the case in the type III algebra of the assumed sort [6, pp. 56—57]. The ideal $\mathscr{M} \cap \mathscr{Z} = \zeta$ is maximal in $\mathscr{Z}$ [10]. Let $[\zeta]$ denote the ideal in $\mathscr{A}$ generated by ζ. Then $\mathscr{M} = \mathscr{J} + [\zeta]$, [6, p. 58, Proposition 2. 3], and $[\zeta]$ is primitive [7, p. 213, Theorem 4. 7]. Furthermore, $\mathscr{J} \notin [\zeta]$, so $\mathscr{M}$ properly contains $[\zeta]$ [6, p. 62]. Thus the result follows by Theorem 2. 3 and Proposition 2. 1.

3. A different approach yields partial results in the type I case.

Theorem 3. 1. *Let $\mathscr{A}$ be a type* I *von Neumann algebra such that $\mathscr{A} = \mathscr{B}(\mathscr{H})\oplus \oplus C(\Omega)$. Let $\mathscr{I}$ be a uniformly closed weakly dense ideal in $\mathscr{A}$ of the form $\mathscr{I} = \mathscr{B}(\mathscr{H})\oplus \oplus \mathscr{J}$, for some ideal $\mathscr{J}$ in $C(\Omega)$. Then $A\in\mathscr{A}$ is thin relative to $\mathscr{I}$ if and only if A satisfies (H).*

Proof. The ideal $\mathscr{J}$ is $\mathscr{J}=\{f\in C(\Omega):f|_\Gamma\equiv 0\}$, for some closed set $\Gamma\subset\Omega$. Since $\mathscr{I}$ is weakly dense, the interior of Γ is empty. Let $\{x_i\}$ be some orthonormal basis for $\mathscr{H}$. Then $A\in\mathscr{A}$ may be written $A=(a_{ij})$, $a_{ij}\in C(\Omega)$.

Suppose $A\in\mathscr{A}$ is not thin. We wish to show $\lim\limits_{P\in\mathscr{P}}\|PAP-AP\|\neq 0$. We claim it suffices to consider $A=(a_{ij})$ with some $a_{rs}\notin\mathscr{J}$, for $r\neq s$. For, suppose $a_{ij}\in\mathscr{J}$, all $i\neq j$. Then $A\neq Z+K$ any $Z\in\mathscr{Z}$, $K\in\mathscr{I}$ implies that for some $v\in\Gamma$, and some indices $r\neq s$, $a_{rr}(v)\neq a_{ss}(v)$. Let $U=(u_{ij})$ be the unitary in $\mathscr{A}$ given by $u_{rr}=u_{rs}=u_{ss}= =1/\sqrt{2}$, $u_{sr}=-1/\sqrt{2}$, and $u_{ii}=1$ if $i\neq r,s$; $u_{ij}=0$ otherwise (all these being constant functions on Ω). It is easy to compute that if $B=U^*AU=(b_{ij})$, then $b_{rs}(v)\neq 0$. Thus $b_{rs}\notin\mathscr{J}$. Observe that $\lim\limits_{P\in\mathscr{P}}\|PAP-AP\| = \lim\limits_{P\in\mathscr{P}}\|PBP-BP\|$, so it suffices to show that this latter limit is non-zero. Thus the claim is established.

Assume $a_{rs}\notin\mathscr{J}$, so fix $\mu\in\Gamma$ with $|a_{rs}(\mu)|>0$. Let $\delta>0$ be a number such that $|a_{rs}(v)|>\delta$, all $v\in V$ some open neighborhood of μ. Choose any $P\in\mathscr{P}$, $P=(p_{ij})$. We construct a projection $Q\in\mathscr{P}$, $Q>P$ with $\|QAQ-AQ\|>\delta/2$. This will suffice to show $\lim\limits_{P\in\mathscr{P}}\|PAP-AP\|\neq 0$.

There is an open neighborhood X of Γ in Ω with $p_{rr}(v)<\varepsilon$, $p_{ss}(v)<\varepsilon$, all $v\in X$. Since the interior of Γ is empty, $V\cap(X\setminus\Gamma)$ is a non-empty open set. Let W be a non-empty open and closed subset of $V\cap(X\setminus\Gamma)$. Define projections in $\mathscr{P}$: $E=(e_{ij})$, $F=(f_{ij})$ with $e_{ss}=\chi_W=f_{rr}$, and $e_{ij}=0=f_{ij}$ otherwise. Set $Q=P\vee E$, so $Q\in\mathscr{P}$ also. If $T=(t_{ij})\in\mathscr{A}$, denote $T(v)=\{t_{ij}(v)\}\in\mathscr{B}(\mathscr{H})$ for each $v\in\Omega$.

Then

$$\|PE\|^2=\sup_{v\in\Omega}\sup_{\substack{x\in E(v)\mathscr{H}\\ \|x\|=1}}(P(v)x,x)=\sup_{v\in W}(P(v)x_s,x_s)=\sup_{v\in W}p_{ss}(v)<\varepsilon.$$

Similarly, $\|PF\|^2<\varepsilon$, and $EF=0$.

Observe that $\|QF\|$ is also small. For, let $v\in\Omega$, and consider $f\in F(v)\mathscr{H}$, $q\in Q(v)\mathscr{H}$, $p\in P(v)\mathscr{H}$ and $e\in E(v)\mathscr{H}$. Then

$$\|Q(v)F(v)\|^2=\sup_{\|f\|=1}\|Q(v)f\|^2=\sup_{\|f\|=1}\sup_{\|q\|=1}|(q,f)|^2.$$

For an arbitrary fixed pair of such vectors q and f, we can write $q=\gamma_1 p+\gamma_2 e$ where p and e are unit vectors. Then we can find some unit vector g with $(e,g)=0$, such that $p=v_1 e+v_2 g$. Using the fact that $\|PE\|^2<\varepsilon$, $\|PF\|^2<\varepsilon$ and $EF=0$, a routine calculation shows that $|(g,f)|^2 < \varepsilon/1-\varepsilon^2$. Thus $\|QF^2\| = \sup_{v\in\Omega} \|Q(v)F(v)\|^2 <$ $< \varepsilon/1-\varepsilon^2$. Therefore,

$$\|(I-Q)AQ\| \geqq \|F(I-Q)AQE\| \geqq \|FAE\| - \|FQAE\| > \sup_{v\in\Omega} \|F(v)A(v)E(v)\| -$$

$$-(\varepsilon/1-\varepsilon^2)^{1/2}\|A\| = \sup_{v\in w} |a_{rs}(v)| - (\varepsilon/1-\varepsilon^2)^{1/2}\|A\| > \delta - (\varepsilon/1-\varepsilon^2)^{1/2}\|A\|.$$

For a sufficiently small choice of ε, $\|(I-Q)AQ\| > \delta/2$.

The converse follows by Proposition 2. 1, and the proof is complete.

Note that if $\mathscr{A}$ is a type I infinite algebra, and $\mathscr{M}$ a maximal ideal, then Theorem 2. 5 characterizes the thin operators relative to $\mathscr{M}$. If $\mathscr{I}$ is a finite intersection of maximal ideals in $\mathscr{A}$, it is not hard to show that an operator satisfying (H) is thin relative to $\mathscr{I}$. For example, if $\mathscr{H}$ is separable and $\mathscr{A} = \mathscr{B}(\mathscr{H}) \oplus C(\Omega)$, then there is a finite set $\Gamma \subset \Omega$ with $\mathscr{I} = \{A \in \mathscr{A} : A(v)$ is compact, all $v \in \Gamma\}$.

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On the invariant subspace lattice $1+\omega^*$

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Abstract: This paper has two parts. In the first one it is shown that a Banach algebra with identity whose lattice of closed left ideals is order isomorphic to $1+\omega^*$ with "dimension gaps" equal to one, from one ideal to the next one, is always a (commutative) Banach algebra of power series with one-point Gelfand spectrum. In the second one, the fact that the algebra $\mathfrak{L}(\mathfrak{X})$ of all bounded linear operators on a complex separable Banach space $\mathfrak{X}$ contains a subalgebra with the above mentioned characteristics, is used to show that $\mathfrak{L}(\mathfrak{X})$ can be generated by two elements.

1. Throughout this paper $\mathfrak{X}$ will denote a complex separable infinite dimensional Banach space. Let $\mathfrak{L}(\mathfrak{X})$ be the set of all operators in $\mathfrak{X}$. Here and in what follows, *operator* will mean *bounded linear map* (from $\mathfrak{X}$ into $\mathfrak{X}$); similarly, *algebra* and *subspace* will mean *weakly closed subalgebra of* $\mathfrak{L}(\mathfrak{X})$ *containing the identity* I of $\mathfrak{X}$ and *closed linear manifold*, respectively. For a given algebra $\mathfrak{A}$, Lat $\mathfrak{A}$ denotes the lattice of invariant (under every operator in $\mathfrak{A}$) subspaces of $\mathfrak{A}$. $\mathfrak{A}$ is called a *strictly cyclic algebra* (s. c. a.) if there exists a vector $x_0\in\mathfrak{X}$ such that

$$\mathfrak{X}=\mathfrak{A}x_0=\{Ax_0 : A\in\mathfrak{A}\}.$$

$\mathfrak{A}$ is separated by $x_0\in\mathfrak{X}$ if $A\in\mathfrak{A}$ and $Ax_0=0$ imply $A=0$. If $\mathfrak{A}x_0=\mathfrak{X}$ and x_0 separates points of $\mathfrak{A}$, then we shall say that $\mathfrak{A}$ is a separated s. c. a. and that x_0 is a separating s. c. vector for $\mathfrak{A}$. It is known that if $\mathfrak{A}$ has a separating s. c. vector x_0, then the map $A\to Ax_0$ from $\mathfrak{A}$ onto $\mathfrak{X}$ is an isomorphism of Banach spaces. By means of this map, $\mathfrak{X}$ can be identified with a Banach algebra $\mathfrak{B}$ with identity e; then $\mathfrak{A}$ is identified with $\mathfrak{B}_L$, the algebra of all left multiplications in $\mathfrak{B}$ by elements of $\mathfrak{B}$ (i.e., the *regular left representation* of $\mathfrak{B}$) and $\mathfrak{A}'$, the commutant of $\mathfrak{A}$ in $\mathfrak{L}(\mathfrak{X})$, is identified with $\mathfrak{B}_R$ the algebra of all right multiplications (or, the *regular right representation* of $\mathfrak{B}$) (see [2; 5; 6; 10]).

Let $\mathfrak{B}$ be a Banach algebra with identity and let $\mathfrak{B}_L$ be its regular left representation; the invariant subspaces of $\mathfrak{B}_L$ are, precisely, the closed left ideals of $\mathfrak{B}$. This justifies the following notation:

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Lat $\mathfrak{B}$ = Lat $\mathfrak{B}_L$ = {closed left ideals of $\mathfrak{B}$}.

The first part of this paper is devoted to proving that a class of Banach algebras with linearly ordered lattice are singly generated.

Theorem 1. *Let $\mathfrak{A}$ be a separated s. c. a. on $\mathfrak{X}$ and assume that*

(1) $\operatorname{Lat}\mathfrak{A} = \{(0)\} \cup \{\mathfrak{M}_n\}_{n=0}^{\infty}$,

where

(2) $\mathfrak{X} = \mathfrak{M}_0 \supset \mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \cdots \supset \mathfrak{M}_n \supset \mathfrak{M}_{n+1} \supset, \ldots,$ *and*

(3) $\dim \mathfrak{M}_n/\mathfrak{M}_{n+1} = 1$, *for all* $n = 0, 1, 2, \ldots$.

Then there exists a quasi-nilpotent operator $T \in \mathfrak{A}$ such that

i) $\mathfrak{M}_n = \text{closure } T^n(\mathfrak{X})$, $n = 0, 1, 2, \ldots$,

ii) $\mathfrak{A} = \mathfrak{A}_T =$ *strong closure of the polynomials in T, and*

iii) $\mathfrak{A}$ *is a Banach algebra of power series in the sense of Lorch and Shilow* (see [8; 10, p. 317; 12]). *In particular, $\mathfrak{A}$ is abelian; the Gelfand spectrum of $\mathfrak{A}$ consists of a single point.*

Remarks. (a) Lat $\mathfrak{A}$ is always complete; hence (1) and (2) imply $\bigcap_{n=0}^{\infty} \mathfrak{M}_n = (0)$.

(b) (2) says that Lat $\mathfrak{A}$ is order isomorphic to $1+\omega^*$ (where where ω is the first non-finite ordinal number). (3) says that the "dimensional gaps" are all equal to one. An invariant subspace (or closed left ideal) lattice satisfying (1), (2) and (3) will be denoted by: $\operatorname{Lat}\mathfrak{A} \cong 1+\omega^*(dg = 1)$.

(c) It was shown in [6] that, if $\mathfrak{A}$ is a separated s. c. a., then the uniform and the strong operator topologies coincide on $\mathfrak{A}$. Hence, in Theorem 1, ii), "strong closure" is actually equivalent to "uniform closure".

Because of the previous identification of $\mathfrak{X}$ with a Banach algebra with identity, Theorem 1 can be rephrased as

Theorem 1′. *Let $\mathfrak{B}$ be a Banach algebra with identity e and assume that* $\operatorname{Lat}\mathfrak{B} \cong 1+\omega^*(dg = 1)$. *Then $\mathfrak{B}$ is a Banach algebra of power series with a quasi-nilpotent generator t. The Gelfand spectrum of $\mathfrak{B}$ consists of a single point and the only non-zero closed left ideals of $\mathfrak{B}$ are those of the form* $\mathfrak{M}_n = \operatorname{cl}(t^n\mathfrak{B})$, $(n = 0, 1, 2, \ldots)$.

2. In what follows, $\mathfrak{B}$ will always denote a Banach algebra with identity e. The proof of Theorem 1 follows from a combination of Banach algebra methods and invariant subspace theory.

Lemma 1. *If* Lat $\mathfrak{B}$ *is linearly ordered, then every closed left ideal is a bilateral ideal.*

Lemma 2. *If* $\operatorname{Lat}\mathfrak{B} \cong 1+\omega^*(dg = 1)$, *then* $\mathfrak{B}$ *has no zero divisors.*

Corollary 3. *Assume that* $\operatorname{Lat}\mathfrak{B} \cong 1+\omega^*(dg = 1)$ *and let* $t\in\mathfrak{B}$, $t\neq 0$. *Then the map*

$$\sum_{k=0}^{N} c_k z^k \to \sum_{k=0}^{N} c_k t^k$$

from the polynomials in one indeterminate into $\mathfrak{B}$ *is one-to-one.*

Lemma 1 follows from a general fact about operator algebras: consider $\operatorname{Lat}\mathfrak{B} = \operatorname{Lat}\mathfrak{B}_L$ under the topology for invariant subspaces given in [7] (see also [1, 11]); since $\operatorname{Lat}\mathfrak{B}_L$ is linearly ordered, every point of $\operatorname{Lat}\mathfrak{B}_L$ is isolated. Therefore, ([7]) $\operatorname{Lat}\mathfrak{B}_L \subset \operatorname{Lat}(\mathfrak{B}_L)' = \operatorname{Lat}\mathfrak{B}_R$. Finally, observe that $\operatorname{Lat}\mathfrak{B}_R = \{\text{closed right ideals of } \mathfrak{B}\}$, from which the result follows.

Now assume that $\operatorname{Lat}\mathfrak{B} \cong 1+\omega^*(dg = 1)$ and let $a, b\in\mathfrak{B}$, $a\neq 0$ and $ab=0$. Then $\mathfrak{M} = \{c\in\mathfrak{B}: cb=0\} = \ker R_b$ (R_b=right multiplication by $b\in\mathfrak{B}_R$) is a non-zero closed left ideal and therefore $\mathfrak{M}=\mathfrak{M}_k$ for some $k\geqq 0$, hence $\dim\mathfrak{B}/\mathfrak{M}_k = k<\infty$. It follows that rank $R_b \leqq k<\infty$.

On the other hand, closure range $R_b = \operatorname{cl}(R_b\mathfrak{B})\in\operatorname{Lat}\mathfrak{B}_L$. Thus, either $\operatorname{cl}(R_b\mathfrak{B}) = (0)$ (and therefore $b=0$) or $\operatorname{cl}(R_b\mathfrak{B}) = \mathfrak{M}_h$, for some $h\geqq 0$. Since $\dim\operatorname{cl}(R_b\mathfrak{B}) = \operatorname{rank} R_b \leqq k<\infty$ and $\dim\mathfrak{M}_h$ is not finite, the second case must be ruled out. We conclude that $b=0$, and the proof of Lemma 2 is complete.

Finally, Corollary 3 is an easy consequence of Lemma 2.

Lemma 4. *Assume that* $\operatorname{Lat}\mathfrak{B} \cong 1+\omega^*(dg = 1)$ *and let* $a\in\mathfrak{B}$. *Then*

i) $\sigma(a)$ (=the spectrum of a in $\mathfrak{B}$) *consists of a single point; moreover,* $\sigma(a) = \sigma(L_a) = \sigma(R_a)$, *where* $\sigma(L_a)$ ($\sigma(R_a)$, *respectively*) *denotes the spectrum of the left (right, respectively) multiplication by a as an operator on* $\mathfrak{B}$.

ii) *a is invertible in $\mathfrak{B}$ if and only if for some $n\geqq 0$ and some* $b\in\mathfrak{M}_n\setminus\mathfrak{M}_{n+1}$, $ab\in\mathfrak{M}_n\setminus\mathfrak{M}_{n+1}$.

Proof. i) Observe that $\mathfrak{B}$ has a *unique maximal bilateral ideal,* $\mathfrak{M}_1$, and that $\dim\mathfrak{B}/\mathfrak{M}_1 = 1$; hence, given $a\in\mathfrak{B}$, there exists a unique complex number $\lambda=\lambda(a)$ such that $a-\lambda e\in\mathfrak{M}_1$. Therefore, $\lambda\in\sigma(a)$.

If $\mu\neq\lambda$, then $\operatorname{cl}(a-\mu e)\mathfrak{B}$ is a closed left ideal of $\mathfrak{B}$, not contained in $\mathfrak{M}_1$. It follows that $\operatorname{cl}(a-\mu e)\mathfrak{B} = \mathfrak{B}$; then $(a-\mu e)\mathfrak{B}$ is a dense left ideal of $\mathfrak{B}$ and therefore (see [10, Chapter 1]) $(a-\mu e)$ has a right inverse b in $\mathfrak{B}$. Since, by Lemma 2, $\mathfrak{B}$ has no zero divisors, $a-\mu e\neq 0$ and $(a-\mu e)[e-b(a-\mu e)] = 0$, we conclude that $b = (a-\mu e)^{-1}$, i.e. $\mu\notin\sigma(a)$. Therefore $\sigma(a) = \{\lambda(a)\}$.

The remaining statements follow from [6; 10] (in particular, a is invertible in $\mathfrak{B}$ if and only if L_a is invertible in $\mathfrak{L}(\mathfrak{B})$ if and only if R_a is invertible in $\mathfrak{L}(\mathfrak{B})$).

ii) If a is invertible, then $a\mathfrak{M}_n\subset\mathfrak{M}_n$ and $a^{-1}\mathfrak{M}_n\subset\mathfrak{M}_n$, for all $n\geqq 0$. It follows that $a\mathfrak{M}_n = a^{-1}\mathfrak{M}_n = \mathfrak{M}_n$ for all n. Therefore, $a(\mathfrak{M}_n\setminus\mathfrak{M}_{n+1}) = \mathfrak{M}_n\setminus\mathfrak{M}_{n+1}$, for all n.

Conversely, if $ab \in \mathfrak{M}_n \setminus \mathfrak{M}_{n+1}$ for some $b \in \mathfrak{M}_n \setminus \mathfrak{M}_{n+1}$ (and some $n \geqq 0$), then we can write $\mathfrak{M}_n = \mathfrak{M}_{n+1} \oplus \{\lambda b : \lambda \in \mathbf{C}\}$ and $ab = \lambda b + b'$, for some $\lambda \neq 0$ and some $b' \in \mathfrak{M}_{n+1}$. Hence, $(a - \lambda e)b = b' \in \mathfrak{M}_{n+1}$. If $(a - \lambda e)$ were invertible, then $b = (a - \lambda e)^{-1} b'$ would belong to $\mathfrak{M}_{n+1}$ (because $\mathfrak{M}_{n+1}$ is invariant under $\mathfrak{B}$!), contradicting our assumption. This proves that $a - \lambda e$ is not invertible in $\mathfrak{B}$. Now, i) implies that $\sigma(a) = \{\lambda\}$; since $\lambda \neq 0$, we conclude that a is invertible.

Proof of Theorem 1'. Let t be any element of $\mathfrak{M}_1 \setminus \mathfrak{M}_2$.

Claim. $\mathfrak{B}$ coincides with the uniform closure of the polynomials in t.

Assume that, for each $n \geqq 0$, $t^n \in \mathfrak{M}_n \setminus \mathfrak{M}_{n+1}$. Then, since $\dim \mathfrak{M}_n / \mathfrak{M}_{n+1} = 1$, for all n, it is not difficult to see that the finite linear combinations of the t^n's, $n = 0, 1, 2, \ldots$ (i.e. the polynomials in t) are uniformly dense in $\mathfrak{B}$.

Thus, in order to prove our claim, we only have to show that $t^n \in \mathfrak{M}_n \setminus \mathfrak{M}_{n+1}$, for $n = 0, 1, 2, \ldots$. We proceed by induction. Our choice of t implies that the above result is true for $n = 0, 1$; let $m > 1$ be the first index such that $t^m \notin \mathfrak{M}_m \setminus \mathfrak{M}_{m+1}$. Since $t^{m-1} \in \mathfrak{M}_{m-1}$, t^{m-1} is not invertible and Lemma 4 implies that $t^m = t^{m-1} t \in \mathfrak{M}_m$; thus, our hypothesis on t^m is equivalent to: $t^m \in \mathfrak{M}_{m+1}$. It follows that $\mathfrak{M} = \{\lambda t^{m-1}\} \oplus \oplus \mathfrak{M}_{m+1} \in \operatorname{Lat} L_t \setminus \operatorname{Lat} \mathfrak{B}$; therefore, there exists $a \in \mathfrak{B}$ such that $a\mathfrak{M} \not\subset \mathfrak{M}$.

Let $b \in \mathfrak{M}_m \setminus \mathfrak{M}_{m+1}$; it is not hard to see that a can be written (in a unique form) as

$$a = \lambda_0 e + \lambda_1 t + \cdots + \lambda_{m-1} t^{m-1} + \lambda_m b + a',$$

where $\lambda_0, \ldots, \lambda_m \in C$ and $a' \in \mathfrak{M}_{m+1}$.

Now, the invariance of $\mathfrak{M}$ under L_t implies that $t^n \mathfrak{M} \subset \mathfrak{M}$, for all $n \geqq 0$. On the other hand, since $a' \in \mathfrak{M}_{m+1}$ and $\mathfrak{M}_{m+1}$ is a bilateral ideal (use Lemma 1), it is not hard to see that

$$a'\mathfrak{M} \subset a'\mathfrak{B} \subset \mathfrak{M}_{m+1} \subset \mathfrak{M}.$$

Thus, $a\mathfrak{M} \not\subset \mathfrak{M}$ if and only if $\lambda_m \neq 0$ and $b\mathfrak{M} \neq \mathfrak{M}$. Moreover, $b\mathfrak{M}_{m+1} \subset \mathfrak{M}_{m+1}$; therefore, $a\mathfrak{M} \not\subset \mathfrak{M}$ is equivalent to: $bt^{m-1} \notin \mathfrak{M}$. But this last statement cannot be true. In fact, since b is not invertible and $t^{m-1} \in \mathfrak{M}_{m-1} \setminus \mathfrak{M}_m$, it follows from Lemma 4 that $bt^{m-1} \in \mathfrak{M}_m$, i.e., $bt^{m-1} = \lambda b + b'$, for some $\lambda \in C$ and some $b' \in \mathfrak{M}_{m+1}$. Now if $\lambda = 0$, then $bt^{m-1} \in \mathfrak{M}_{m+1} \subset \mathfrak{M}$, contradicting our assumption. If $\lambda \neq 0$, then $b(t^{m-1} - \lambda e) = b' \in \mathfrak{M}_{m+1}$ and (by Lemma 2) $(t^{m-1} - \lambda e)$ is invertible in $\mathfrak{B}$; hence $b = b'(t^{m-1} - \lambda e)^{-1} \in \mathfrak{M}_{m+1}$ (here we are using the fact that $\mathfrak{M}_{m+1}$ is a bilateral ideal, i.e., Lemma 1), again we obtain a contradiction.

We conclude that $t^n \in \mathfrak{M}_n \setminus \mathfrak{M}_{n+1}$, for all $n \geqq 0$ and $\mathfrak{B}$ is the uniform closure of the polynomials in t.

By Lemma 4, t is quasi-nilpotent. To complete the proof we only have to show that $\mathfrak{B}$ is a Banach algebra of power series in t. This is also clear: observe that, for each fixed $n \geqq 0$, $\mathfrak{B}$ can be written (in a unique fashion) as the direct sum

$$\mathfrak{B} = \{\lambda_0 e\} \oplus \{\lambda_1 t\} \oplus \cdots \oplus \{\lambda_n t^n\} \oplus \mathfrak{M}_{n+1}$$

with complex $\lambda_0, \ldots, \lambda_n$.

Define γ_k by $\gamma_k(t^n)=1$, $\gamma_k(t^k)=0$ for $n = 0, 1, \ldots, k-1$ and $\gamma_k(\mathfrak{M}_{k+1})=0$. It is clear that γ_k is a continuous linear functional on $\mathfrak{B}$ and that every element a of $\mathfrak{B}$ can be written as a (unique!) formal power series in t: $a = \sum_{k=0}^{\infty} \gamma_k(a) t^k$. Since t generates $\mathfrak{B}$ and t is quasi-nilpotent, it is not hard to infer that the only non-zero continuous multiplicative functional on $\mathfrak{B}$ is γ_0; i.e. Gelfand's spectrum of $\mathfrak{B}$ is a single point.

It is completely apparent that $\mathfrak{M}_n = \mathrm{cl}\,(t^n \mathfrak{B})$; $n=0, 1, 2, \ldots$. The proof is complete now.

3. Generators of $\mathfrak{L}(\mathfrak{X})$. Recently, S. GRABINER ([3; 4]) showed that, for any $\mathfrak{X}$ satisfying our requirements it is possible to construct a chain $\mathfrak{J} \cong 1+\omega^*(dg = 1)$ of subspaces and a nuclear operator T such that

(4) T is a quasi-nilpotent; $\mathfrak{A}_T$ is a separated s. c. a., and $\mathrm{Lat}\, T = \mathfrak{J}$.

We are indebted to Professor GRABINER for sending us his unpublished paper [4] and to M. IMINA for several helpful discussions. We shall use Grabiner's result to prove that $\mathfrak{L}(\mathfrak{X})$ is always generated by two elements. In fact, we have the following:

Theorem 2. *Let $L \in \mathfrak{L}(\mathfrak{X})$, $L \neq \lambda I$ (for all $\lambda \in C$). Then there exists $T \in \mathfrak{L}(\mathfrak{X})$ such that $\mathfrak{L}(\mathfrak{X}) = \mathfrak{A}(T, L)$, the strong closure of the polynomials in T and L.*

The construction of a chain $\mathfrak{J} \cong 1+\omega^*(dg=1)$ of subspaces of $\mathfrak{X}$ is standard. This is equivalent to finding a sequence $\{\beta_n\}_{n=0}^{\infty} \subset \mathfrak{X}^*$ (the topological dual of $\mathfrak{X}$) such that the β_n' s are linearly independent, $\mathfrak{M}_n = \bigcap_{k=0}^{n-1} \ker \beta_k$ $(n=1, 2, 3, \ldots)$ and $\bigcap_{k=0}^{\infty} \ker \beta_k = (0)$ (i.e., $\{\beta_n\}$ is *total* on $\mathfrak{X}$). Then, if $\mathfrak{M}_0 = \mathfrak{X}$, the lattice $\mathfrak{J} = \{(0)\} \cup \{\mathfrak{M}_n\}$ satisfies our requirements; furthermore, $\beta_0 (\neq 0)$ can be arbitrarily chosen in $\mathfrak{X}^*$. We shall need two auxiliary lemmas; the first one says that a "small perturbation" of $\{\beta_n\}$ provides a new lattice, $\mathfrak{J}'$ with similar characteristics.

Lemma 5. *Let $\{\beta_n\}_{n=0}^{\infty}$ be a total set of linearly independent functionals such that $\bigcap_{n=0}^{\infty} \ker \beta_n = (0)$ and $\|\beta_n\| \geqq 1$ for all $n \geqq 0$. If $0 \leqq \varepsilon_n \leqq (2\|\beta_n\|)^{-1}$, then $\{\beta_n' = \beta_n + \varepsilon_{n+1} \beta_{n+1}\}_{n=0}^{\infty}$ is also a linearly independent total set of linear functionals on $\mathfrak{X}$.*

Proof. Let $x\in\mathfrak{X}$ and assume that $\beta_n'(x)=0$, for all n. Then $\beta_n(x)=-\varepsilon_{n+1}\beta_{n+1}(x)$ and, by induction on k, we have

$$\beta_n(x)=(-1)^k\varepsilon_{n+1}\varepsilon_{n+2}\dots\varepsilon_{n+k-1}\varepsilon_{n+k}\beta_{n+k}(x), \qquad k=2, 3, \dots .$$

Hence

$$|\beta_n(x)|\leqq 2^{-(k-1)}\|\varepsilon_{n+k}\beta_{n+k}\|\,\|x\|\leqq 2^{-(k-1)}\|x\|\to 0, \quad \text{as} \quad k\to\infty.$$

Therefore $\beta_n(x)=0$ for all $n\geqq 0$. Since $\{\beta_n\}_{n=0}^\infty$ is total, it follows that $x=0$; i.e., $\{\beta_n'\}$ is also a total set. The linear independence of $\{\beta_n'\}$ is also clear.

Lemma 6. *Let $L\in\mathfrak{L}(\mathfrak{X})$, $L\neq\lambda L$. Then there exists a lattice $\mathfrak{J}\cong 1+\omega^*(dg=1)$ such that* $\operatorname{Lat} L\cap\mathfrak{J}=\{(0),\mathfrak{X}\}$.

Proof. If every $\beta\in\mathfrak{X}^*$ is an eigenvector of L^*, then $L^*=\lambda I^*$ (I^* = the identity operator on $\mathfrak{X}^*$), for some $\lambda\in C$, and therefore $L=\lambda I$, contradicting our hypothesis. Therefore, we can find a vector $\beta_0\in\mathfrak{X}^*$, $\|\beta_0\|\geqq 1$. which is not an eigenvector of L^*; equivalently, $\ker\beta_0=\mathfrak{M}_1\notin\operatorname{Lat} L$.

Complete $\{\beta_0\}$ to a total set $\{\beta_n\}_{n=0}^\infty$ of linearly independent functionals of norm $\geqq 1$ and set $\beta_n'=\beta_n+\varepsilon_{n+1}\beta_{n+1}$, where $\varepsilon_n=(2\|\beta_n\|)^{-1}$, $n=0, 1, 2, \dots$.

Write $\mathfrak{L}=(0^*)$, $\mathfrak{L}_1=\{\lambda\beta_0:\lambda\in C\}$. Now we proceed by induction; assume that $\mathfrak{L}_n$ has been defined in such a way that $\mathfrak{L}_n\subset\operatorname{lin\,span}[\{\beta_0,\dots,\beta_n\}$, $\mathfrak{L}_n\notin\operatorname{Lat} L^*$ and $\beta_{m-1}\notin\mathfrak{L}_{m-1}$ for $n=0, 1, \dots, m-1$. If both $\mathfrak{L}_{m-1}\oplus\{\lambda\beta_{m-1}\}$ and $L_{m-1}\oplus\{\lambda\beta_{m-1}'\}$ belong to $\operatorname{Lat} L^*$, then $\mathfrak{L}_{m-1}=(\mathfrak{L}_{m-1}\oplus\{\lambda\beta_{m-1}\})\cap(\mathfrak{L}_{m-1}\oplus\{\lambda\beta_{m-1}'\})\in\operatorname{Lat} L^*$, contradicting our inductive hypothesis. Hence, either $\mathfrak{L}_{m-1}\oplus\{\lambda\beta_{m-1}\}\notin\operatorname{Lat} L^*$ or $\mathfrak{L}_{m-1}\oplus\oplus\{\lambda\beta_{m-1}\}\in\operatorname{Lat} L^*$ and $\mathfrak{L}_{m-1}\oplus\{\lambda\beta_{m-1}'\}\notin\operatorname{Lat} L^*$. In the first case, we write $\mathfrak{L}_m=\mathfrak{L}_{m-1}\oplus\oplus\{\lambda\beta_{m-1}\}$ and $\beta_{m-1}''=\beta_{m-1}$; in the second one, we take $\mathfrak{L}_m=\mathfrak{L}_{m-1}\oplus\{\lambda\beta_{m-1}'\}$ and $\beta_{m-1}''=\beta_{m-1}'$.

Thus we have constructed a sequence $\{\mathfrak{L}_n\}_{n=0}^\infty$ of subspaces such that 1) $\mathfrak{L}_n\subset\mathfrak{L}_{n+1}$; 2) $\dim\mathfrak{L}_n=n$, $n=0, 1, \dots$, and 3) $\mathfrak{L}_n\notin\operatorname{Lat} L^*$. It is not hard to see using Lemma 5 that the lattice $\mathfrak{J}=\{(0)\}\cup\{\mathfrak{M}_n\}_{n=0}^\infty$ where $\mathfrak{M}_n=\mathfrak{L}_n^\perp=\bigcap_{k=0}^{n-1}\ker\beta_k''$ satisfies our requirements, i.e. $\mathfrak{J}\cong 1+\omega^*(dg=1)$ and $\operatorname{Lat} L\cap\mathfrak{J}=\{(0),\mathfrak{X}\}$.

Now we are in a position to prove Theorem 2. Let $L\in\mathfrak{L}(\mathfrak{X})$, $L\neq\lambda I$ and let $\mathfrak{J}$ be chosen as in Lemma 6. Using Grabiner's result, we construct an operator $T\in\mathfrak{L}(\mathfrak{X})$ satisfying (4). It is clear that $\mathfrak{A}(T, L)$ is a *transitive* subalgebra of $\mathfrak{L}(\mathfrak{X})$ (i.e., it has no non-trivial invariant subspaces); in fact,

$$\operatorname{Lat}\mathfrak{A}(T, L)=\operatorname{Lat} L\cap\operatorname{Lat} T=\operatorname{Lat} L\cap\mathfrak{J}=\{(0),\mathfrak{X}\}.$$

On the other hand, $\mathfrak{A}_T$ is a strictly cyclic subalgebra of $\mathfrak{A}(T, L)$. These two properties of $\mathfrak{A}(T, L)$ and the results of [2; 5] imply that $\mathfrak{A}(T, L)=\mathfrak{L}(\mathfrak{X})$.

Remarks. a) In [9], H. Radjavi and P. Rosenthal proved that if L is any operator in the complex (or real) separable Hilbert space $\mathfrak{X}$ such that $L\neq\lambda I$ ($\lambda\in C$), then there exists a compact hermitian operator $H \in \mathfrak{L}(\mathfrak{X})$ such that $\mathfrak{L}(\mathfrak{X}) = \mathfrak{A}(L, H)$. Thus, Theorem 2 can be considered as a result for Banach spaces which is analogous to the above one.

b) Let $\mathfrak{X}$ be as usual and let β be a non-zero continuous linear functional on $\mathfrak{X}$. For each $z \in \ker \beta$ and each $\lambda \in C$, define $A_{z,\lambda} y = \lambda y + \beta(y) z$ and $\mathfrak{A} = \{A_{z,\lambda} : z \in \ker \beta, \lambda \in C\}$; it is not hard to check (see [2]) that $\mathfrak{A}$ is an abelian separated s. c. a. (x_0 is a separating s. c. vector for $\mathfrak{A}$ if and only if $\beta(x_0) \neq 0$). A straightforward computation shows that the subalgebra generated by $\{A_{z_\nu}, \lambda_\nu : \nu \in \Sigma\}$ is equal to $\{A_{z,\lambda} : \lambda \in C, z \in \text{closed lin span } [z_\nu : \nu \in \Sigma]\}$; in particular, $\mathfrak{A}$ cannot be finitely generated.

With minor modifications of the same example it is not hard to show that $\mathfrak{L}(\mathfrak{X})$ contains, for each n ($n = 2, 3, 4, \ldots, \aleph_0$), an abelian separated s. c. a. $\mathfrak{A}_n$ which can be generated by n operators, but no set of $n-1$ operators generates $\mathfrak{A}_n$. Thus, the statement of Theorem 2 cannot be extended to arbitrary subalgebras of $\mathfrak{L}(\mathfrak{X})$.

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On the structure of intertwining operators

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A theorem proved in our previous paper [6] asserts that every operator X intertwining two contractions, T_1 and T_2, can be lifted, without increasing norm, to an operator Y intertwining their minimal isometric dilations, V_1 and V_2. This theorem allows a study of the structure of such operators: this will be done, in a purely geometric manner, in Sec. 1. Then, in Sec. 2, the results of Sec. 1 will be reformulated for the case where the contractions T_k $(k=1, 2)$ are completely non-unitary and appear in their functional models $S(\Theta_k)$.

Particular interest lies with intertwining operators X which have a (bounded) inverse and thus establish similarity between T_1 and T_2. We obtain in this way among others a criterion for a contraction to be similar to some isometry (and a new proof of the known criterion for a contraction to be similar to some unitary operator). The main criteria of similarity concern two contractions, arbitrary or completely non-unitary, in the latter case given by their functional models $S(\Theta_k)$ $(k=1, 2)$. One of these criteria, stated in Sec. 3, is particularly interesting since it only involves relations between analytic functions and a certain equidimensionality condition. This criterion generalizes a former result of KRIETE [3], which concerns operators $S(\Theta_k)$ with scalar valued contractive analytic functions Θ_k.

Sec. 4 is devoted to problems concerning the commutant $(T)'$ of a c.n.u. contraction $T=S(\Theta)$. Namely, a necessary condition is given for the characteristic function $\Theta(z)$ in order that $(T)'$ should consist of functions $\varphi(T)$, φ belonging to the Nevanlinna class N_T. Moreover, it is proved that if $T=S(\Theta)$ with scalar Θ, then $(T)'$ is always commutative, with the exception of a single case.

Finally, in Sec. 5, functions $u(T)$ (with $u\in H^\infty$) of a c.n.u. contraction T are considered, and a criterion is established for $u(T)$ to be boundedly invertible; this criterion generalizes an earlier result of FUHRMANN [1].

1. Contractions of general type

1. For any two operators on Hilbert spaces, say T_1 on $\mathfrak{H}_1$ and T_2 on $\mathfrak{H}_2$, denote by $\mathcal{I}(T_1, T_2)$ the set of (linear, bounded) operators $X: \mathfrak{H}_1 \to \mathfrak{H}_2$ such that

$$T_2 X = X T_1. \tag{1.1}$$

If T_1 and T_2 are *contractions*, and V_1 and V_2 are their minimal isometric dilations (cf. [5], Chapter I) acting on the spaces $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively, then let $\mathcal{I}^+(T_1, T_2)$ denote the set of operators $Y: \mathfrak{R}_1 \to \mathfrak{R}_2$ belonging to $\mathcal{I}(V_1, V_2)$ and satisfying the additional condition

$$P_2 Y P_1 = P_2 Y, \tag{1.2}$$

where P_i denotes the orthogonal projection from $\mathfrak{R}_i$ onto $\mathfrak{H}_i$.

Clearly $\mathcal{I}(T_1, T_2)$ and $\mathcal{I}^+(T_1, T_2)$ are subspaces of the Banach spaces of all operators from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ and from $\mathfrak{R}_1$ into $\mathfrak{R}_2$, respectively.

As T_i and V_i are connected by the relation

$$T_i P_i = P_i V_i, \tag{1.3}$$

condition $Y \in \mathcal{I}^+(T_1, T_2)$ implies

$$T_2 P_2 Y = P_2 V_2 Y = P_2 Y P_1 V_1 = P_2 Y T_1 P_1,$$

i.e. the operator

$$X = P_2 Y | \mathfrak{H}_1 \tag{1.4}$$

belongs to $\mathcal{I}(T_1, T_2)$. Thus the transformation $Y \to X$ defined by (1.4) is a map

$$\pi_{12}: \mathcal{I}^+(T_1, T_2) \to \mathcal{I}(T_1, T_2),$$

which is obviously linear and does not increase norm (i.e., $\|X\| \leqq \|Y\|$). Observe that on account of (1.2) relation (1.4) implies

$$X P_1 = P_2 Y; \tag{1.5}$$

conversely, relation (1.5) implies both (1.2) and (1.4).

The "Lifting Theorem" for intertwining operators (see [6], or [5], Sec. II. 2) asserts that the above map π_{12} is actually *onto*, moreover for every $X \in \mathcal{I}(T_1, T_2)$ there exists at least one $Y \in \mathcal{I}^+(T_1, T_2)$ satisfying (1.5) and such that $\|X\| = \|Y\|$.

The aim of this paper is a further analysis of this map π_{12}, and some of its applications.

To begin with, let us state the following immediate consequence of relation (1.5):

Multiplication Property: If T_1, T_2, T_3 are any three contractions and if

$$Y \in \mathscr{I}^+(T_1, T_2), \quad Z \in \mathscr{I}^+(T_2, T_3),$$

then

(1.6) $$ZY \in \mathscr{I}^+(T_1, T_3) \quad \text{and} \quad \pi_{13}(ZY) = \pi_{23}(Z)\pi_{12}(Y).$$

Also note that

(1.7) $$I_{\mathfrak{K}_1} \in \mathscr{I}^+(T_1, T_1) \quad \text{and} \quad \pi_{11}(I_{\mathfrak{K}_1}) = I_{\mathfrak{H}_1}.$$

2. Let us return to the case of two contractions, T_1 and T_2. Consider the Wold decomposition of the space $\mathfrak{K}_i$ generated by the minimal isometric dilation V_i of T_i $(i=1, 2)$, i.e. let

(1.8) $$\mathfrak{K}_i = \mathfrak{S}_{*i} \oplus \mathfrak{R}_i, \quad \text{where} \quad \mathfrak{R}_i = \bigcap_{n=0}^{\infty} V_i^n \mathfrak{K}_i;$$

the subspaces $\mathfrak{S}_{*i}$ and $\mathfrak{R}_i$ reduce V_i respectively to its unilateral shift part S_{*i} and its unitary part R_i (one of these subspaces may be missing, i.e. equal $\{0\}$). Then we have for any $Y \in \mathscr{I}(V_1, V_2)$:

$$Y\mathfrak{R}_1 = \bigcap_{n=0}^{\infty} YV_1^n \mathfrak{K}_1 = \bigcap_{n=0}^{\infty} V_2^n Y\mathfrak{K}_1 \subset \bigcap_{n=0}^{\infty} V_2^n \mathfrak{K}_2 = \mathfrak{R}_2.$$

Therefore, if both $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are decomposed according to (1.8) the operator Y will be represented by a matrix

(1.9) $$Y = \begin{bmatrix} A_* & 0 \\ B & C \end{bmatrix},$$

where

(1.10) $$A_* \in \mathscr{I}(S_{*1}, S_{*2}), \quad B \in \mathscr{I}(S_{*1}, R_2), \quad C \in \mathscr{I}(R_1, R_2).$$

Clearly, conditions (1.10) are also sufficient for Y to belong to $\mathscr{I}(V_1, V_2)$.

Now we are going to analyse condition (1.2). To this end first recall (*cf.* [5], Sec. II. 2) that the subspace

(1.11) $$\mathfrak{S}_i = \mathfrak{K}_i \ominus \mathfrak{H}_i$$

is invariant for V_i and that

(1.12) $$S_i = V_i | \mathfrak{S}_i$$

is a unilateral shift. (It may happen that $\mathfrak{S}_i = \{0\}$: this is the case if T_i itself is an isometry.)

Introduce the operators

(1.13) $$\left.\begin{matrix} \hat{\Theta}_i \\ \hat{\Delta}_i \end{matrix}\right\} = \text{orthogonal projection of } \mathfrak{S}_i \text{ into} \left\{\begin{matrix} \mathfrak{S}_{*i} \\ \mathfrak{R}_i \end{matrix}\right. .$$

As $\mathfrak{S}_{*i}$ and $\mathfrak{R}_i$ are reducing subspaces for the isometry V_i we obviously have

(1.14) $$\hat{\Theta}_i \in \mathscr{I}(S_i, S_{*i}), \quad \hat{\Delta}_i \in \mathscr{I}(S_i, R_i).$$

Condition (1. 2) means that Y transforms $\mathfrak{S}_1$ into $\mathfrak{S}_2$. Hence we infer that an operator $Y\in\mathcal{I}(V_1, V_2)$ satisfies condition (1. 2) if and only if $A=Y|\mathfrak{S}_1$ belongs to $\mathcal{I}(S_1, S_2)$. Using for Y the matrix form (1. 9) and for $x_i\in\mathfrak{S}_i$ the column vector representation

$$x_i=\begin{bmatrix}\hat{\Theta}_i x_i\\ \hat{\Delta}_i x_i\end{bmatrix}\quad (i=1, 2),$$

and comparing the corresponding components we arrive at the following result:

Lemma 1. 1. *The operator Y with the matrix* (1. 9) *belongs to $\mathcal{I}^+(T_1, T_2)$ if and only if its entries satisfy conditions* (1. 10) *and*

$$A_*\hat{\Theta}_1=\hat{\Theta}_2 A, \quad B\hat{\Theta}_1+C\hat{\Delta}_1=\hat{\Delta}_2 A, \tag{1.15}$$

with some operator

$$A\in\mathcal{I}(S_1, S_2). \tag{1.16}$$

3. Consider a $Y\in\mathcal{I}^+(T_1, T_2)$ for which $\pi_{12}(Y)=0$, i.e. $Y\mathfrak{H}_1\subset\mathfrak{H}_2^\perp(=\mathfrak{S}_2)$. As by virtue of (1. 2) we also have $Y\mathfrak{S}_1\subset\mathfrak{S}_2$, condition $Y\mathfrak{H}_1\subset\mathfrak{S}_2$ is equivalent to the condition $Y\mathfrak{R}_1\subset\mathfrak{S}_2$. Hence we infer first that

$$Y\mathfrak{R}_1=\bigcap_{n=0}^{\infty} YV_1^n\mathfrak{R}_1=\bigcap_{n=0}^{\infty} V_2^n Y\mathfrak{R}_1\subset\bigcap_{n=0}^{\infty} V_2^n\mathfrak{S}_2=\{0\}$$

(the latter equation holds because $V_2|\mathfrak{S}_2(=S_2)$ is a unilateral shift); as a consequence we have $C=0$. Next, $Y\mathfrak{R}_1\subset\mathfrak{S}_2$ also implies $Y\mathfrak{S}_{*1}\subset\mathfrak{S}_2$, and hence we deduce that the operator $D=Y|\mathfrak{S}_{*1}$ belongs to $\mathcal{I}(S_{*1}, S_2)$. Therefore we have for $x\in\mathfrak{S}_{*1}$

$$\begin{bmatrix}A_* & 0\\ B & C\end{bmatrix}\begin{bmatrix}x\\ 0\end{bmatrix}=Yx=Dx=\begin{bmatrix}\hat{\Theta}_2 Dx\\ \hat{\Delta}_2 Dx\end{bmatrix}, \text{ i. e. } A_*=\hat{\Theta}_2 D, \quad B=\hat{\Delta}_2 D.$$

Conversely, one easily verifies that if D is any operator satisfying

$$D\in\mathcal{I}(S_{*1}, S_2) \tag{1.17}$$

then the operators defined by

$$A=D\hat{\Theta}_1, \quad A_*=\hat{\Theta}_2 D, \quad B=\hat{\Delta}_2 D, \quad C=0 \tag{1.18}$$

satisfy conditions (1. 10), (1. 15), and (1. 16), and therefore the corresponding operator $Y=\begin{bmatrix}A_* & 0\\ B & C\end{bmatrix}$ belongs to $\mathcal{I}^+(T_1, T_2)$. Moreover we have then

$$Y\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\hat{\Theta}_2 Dx\\ \hat{\Delta}_2 Dx\end{bmatrix}=Dx\in\mathfrak{S}_2 \quad\text{for}\quad \begin{bmatrix}x\\ y\end{bmatrix}\in\mathfrak{R}_1,$$

and therefore

$$\pi_{12}(Y)=0.$$

Thus we have proved:

Lemma 1. 2. *The general form of an operator* $Y \in \mathscr{I}^+(T_1, T_2)$ *satisfying* $\pi_{12}(Y)=0$ *is*

$$Y=\begin{bmatrix}\hat{\Theta}_2 D & 0\\ \hat{\Delta}_2 D & 0\end{bmatrix} \quad \textit{with arbitrary} \quad D \in \mathscr{I}(S_{*1}, S_2).$$

4. Suppose we have

$$Y=\begin{bmatrix}A_* & 0\\ B & C\end{bmatrix} \in \mathscr{I}^+(T_1, T_2) \quad \text{and} \quad Y'=\begin{bmatrix}A'_* & 0\\ B' & C'\end{bmatrix} \in \mathscr{I}^+(T_2, T_1).$$

Let $X=\pi_{12}(Y)$, $X'=\pi_{21}(Y')$. From the multiplication property (1. 6) and from (1. 7) we deduce that X and X' are inverse to each other if and only if

$$\pi_{11}(I_{\mathfrak{K}_1}-Y'Y)=0 \quad \text{and} \quad \pi_{22}(I_{\mathfrak{K}_2}-YY')=0.$$

On account of Lemmas 1. 1 and 1. 2 these two conditions in turn are equivalent to the condition that there exist operators

$$D \in \mathscr{I}(S_{*1}, S_1), \quad D' \in \mathscr{I}(S_{*2}, S_2)$$

satisfying the equations

$$\begin{bmatrix}I_{\mathfrak{S}_{*1}}-A'_*A_* & 0\\ -B'A_*-C'B & I_{\mathfrak{R}_1}-C'C\end{bmatrix}=\begin{bmatrix}\hat{\Theta}_1 D & 0\\ \hat{\Delta}_1 D & 0\end{bmatrix},$$

$$\begin{bmatrix}I_{\mathfrak{S}_{*2}}-A_*A'_* & 0\\ -BA'_*-CB' & I_{\mathfrak{R}_2}-CC'\end{bmatrix}=\begin{bmatrix}\hat{\Theta}_2 D' & 0\\ \hat{\Delta}_2 D' & 0\end{bmatrix};$$

thus in particular $C'=C^{-1}$. Since $C \in \mathscr{I}(R_1, R_2)$, this implies that the unitary operators R_1, R_2 are similar, and therefore unitarily equivalent.

Note that the existence of a boundedly invertible X in $\mathscr{I}(T_1, T_2)$ means that T_1 and T_2 are similar. Thus, also using Lemma 1. 2, we can summarize our results as follows:

Theorem 1. 3. *A* necessary *condition for the contractions* T_1 *and* T_2 *to be similar is that the unitary parts* R_1, R_2 *of their minimal isometric dilations be unitarily equivalent.* Necessary *and* sufficient *is the existence of operators*

$$(\sigma)\begin{cases}(\sigma_1)\begin{cases}A_* \in \mathscr{I}(S_{*1}, S_{*2}), & A \in \mathscr{I}(S_1, S_2), \quad D \in \mathscr{I}(S_{*1}, S_1)\\ A'_* \in \mathscr{I}(S_{*2}, S_{*1}), & A' \in \mathscr{I}(S_2, S_1), \quad D' \in \mathscr{I}(S_{*2}, S_2),\end{cases}\\[2ex] (\sigma_2)\begin{cases}B \in \mathscr{I}(S_{*1}, R_2), & C \in \mathscr{I}(R_1, R_2),\\ B' \in \mathscr{I}(S_{*2}, R_1), & C' \in \mathscr{I}(R_2, R_1)\end{cases}\end{cases}$$

satisfying the conditions

$$\begin{array}{llll}
(\alpha) & A_*\hat{\Theta}_1=\hat{\Theta}_2 A, & (\beta) & B\hat{\Theta}_1+C\hat{\Delta}_1=\hat{\Delta}_2 A,\\
(\alpha') & A'_*\hat{\Theta}_2=\hat{\Theta}_1 A', & (\beta') & B'\hat{\Theta}_2+C'\hat{\Delta}_2=\hat{\Delta}_1 A',\\
(\gamma_*) & A'_*A_*+\hat{\Theta}_1 D=I_{\mathfrak{S}_{*1}}, & (\delta) & B'A_*+C'B=-\hat{\Delta}_1 D,\\
(\gamma'_*) & A_*A'_*+\hat{\Theta}_2 D'=I_{\mathfrak{S}_{*2}}, & (\delta') & BA'_*+CB'=-\hat{\Delta}_2 D',\\
(\varepsilon) & C'=C^{-1}. & &
\end{array}$$

5. From conditions (α)—(ε) we deduce some further ones. Namely we have

$$\hat{\Theta}_1(A'A+D\hat{\Theta}_1-I_{\mathfrak{S}_1})\overset{\alpha',\gamma_*}{=\!=}A'_*\hat{\Theta}_2A+(I_{\mathfrak{S}_{*1}}-A'_*A_*)\hat{\Theta}_1-\hat{\Theta}_1=A'_*(\hat{\Theta}_2A-A_*\hat{\Theta}_1)\overset{\alpha}{=\!=}0,$$

$$\hat{\Delta}_1(A'A+D\hat{\Theta}_1-I_{\mathfrak{S}_1})\overset{\beta',\delta}{=\!=}(B'\hat{\Theta}_2+C'\hat{\Delta}_2)A-(B'A_*+C'B)\hat{\Theta}_1-\hat{\Theta}_1=$$

$$=B'(\hat{\Theta}_2A-A_*\hat{\Theta}_1)+C'(\hat{\Delta}_2A-B\hat{\Theta}_1)-\hat{\Delta}_1\overset{\alpha,\beta}{=\!=}C'C\hat{\Delta}_1-\hat{\Delta}_1=0,$$

and therefore

$$(\gamma)\qquad A'A+D\hat{\Theta}_1=I_{\mathfrak{S}_1}.$$

By analogous reasons, we have

$$(\gamma')\qquad AA'+D'\hat{\Theta}_2=I_{\mathfrak{S}_2}.$$

Furthermore,

$$\hat{\Theta}_2(AD-D'A_*)\overset{\alpha',\gamma'_*}{=\!=}A_*\hat{\Theta}_1D-(I_{\mathfrak{S}_{*2}}-A_*A'_*)A_*=A_*(\hat{\Theta}_1D+A'_*A_*)-A_*\overset{\gamma_*}{=\!=}0,$$

$$\hat{\Delta}_2(AD-D'A_*)\overset{\beta,\delta'}{=\!=}(B\hat{\Theta}_1+C\hat{\Delta}_1)D+(BA'_*+CB')A_*=$$

$$=B(\hat{\Theta}_1D+A'_*A_*)+C(\hat{\Delta}_1D+B'A_*)\overset{\gamma_*,\delta}{=\!=}B-CC'B\overset{\varepsilon}{=\!=}0,$$

and therefore

$$(\eta)\qquad AD=D'A'_*.$$

By analogous reasons,

$$(\eta')\qquad A'D'=DA_*.$$

Conversely, if the operators occurring in (σ) except B' satisfy conditions (α)—(η') except (β'), (δ), (δ'), then the operator B' defined by

$$B'=-C'(\hat{\Delta}_2D'+BA'_*)$$

will obviously satisfy conditions $B'\in\mathcal{I}(S_{*2},R_1)$ and (δ'); let us show that it also satisfies (β') and (δ). Indeed, we have

$$B'\hat{\Theta}_2+C'\hat{\Delta}_2=-C'\hat{\Delta}_2D'\hat{\Theta}_2-C'BA'_*\hat{\Theta}_2+C'\hat{\Delta}_2=$$

$$\overset{\alpha',\gamma'}{=\!=}-C'\hat{\Delta}_2(I_{\mathfrak{S}_2}-AA')-C'B\hat{\Theta}_1A'+C'\hat{\Delta}_2=$$

$$=C'(\hat{\Delta}_2A-B\hat{\Theta}_1)A'\overset{\beta}{=\!=}C'C\hat{\Delta}_1A'\overset{\varepsilon}{=\!=}\hat{\Delta}_1A',$$

$$B'A_* + C'B = -C'\hat{\Delta}_2 D'A_* - C'BA'_*A_* + C'B =$$
$$\overset{\gamma_*,\eta}{=} -C'\hat{\Delta}_2 AD - C'B(I_{\mathfrak{S}_{*1}} - \hat{\Theta}_1 D) + C'B =$$
$$= -C'(\hat{\Delta}_2 A - B\hat{\Theta}_2)D \overset{\beta}{=} -C(C'\hat{\Delta}_1)D \overset{\varepsilon}{=} -\hat{\Delta}_1 D.$$

Thus we have:

Theorem 1. 3′. *The contractions T_1, T_2 are similar if and only if there exist operators A_*, A'_*, A, A', D, D', C, C' and B satisfying conditions (σ) and (α), (α'), (β), (γ_*), (γ'_*), (γ), (γ'), (η), (η').*

6. Consider the particular case that T_2 is an *isometry*. Then $\mathfrak{R}_2 = \mathfrak{H}_2$, $\mathfrak{S}_2 = \{0\}$; thus A and D (whose *ranges* are in $\mathfrak{S}_2$) as well as A', $\hat{\Theta}_2$ and $\hat{\Delta}_2$ (which are *defined* on $\mathfrak{S}_2$) are all zero operators. Hence conditions (α)—(η') occurring in Theorem 1. 3′ reduce to the following ones:

$(\alpha)_0$ $A_*\hat{\Theta}_1 = 0$,	$(\beta)_0$ $B\hat{\Theta}_1 + C\hat{\Delta}_1 = 0$,
$(\gamma_*)_0$ $A'_*A_* + \hat{\Theta}_1 D = I_{\mathfrak{S}_{*2}}$,	$(\varepsilon)_0$ C: boundedly invertible,
$(\gamma'_*)_0$ $A_*A'_* = I_{\mathfrak{S}_{*2}}$,	$(\eta')_0$ $DA'_* = 0$.
$(\gamma)_0$ $D\hat{\Theta}_1 = I_{\mathfrak{S}_1}$,	

Thus in particular the existence of $D \in \mathscr{I}(S_{*1}, S_1)$ satisfying $(\gamma)_0$ is a necessary condition for T_1 to be similar to *some* isometry. This condition turns out to be also sufficient.

To this effect first observe that by account of relation $D \in \mathscr{I}(S_{*1}, S_1)$ the null-space $\mathfrak{D} = \ker D$ is invariant for S_{*1}. As S_{*1} is a unilateral shift so is $S_{*1}|\mathfrak{D}$ (possibly of multiplicity 0). Consider now the isometry

$$T_2 = (S_{*1}|\mathfrak{D}) \oplus R_1 \quad \text{on} \quad \mathfrak{H}_2 = \mathfrak{D} \oplus \mathfrak{R}_1.$$

Then, clearly

$$\mathfrak{S}_{*2} = \mathfrak{D}, \quad \mathfrak{R}_2 = \mathfrak{R}_1, \quad S_{*2} = S_{*1}|\mathfrak{D}, \quad \text{and} \quad R_2 = R_1.$$

Set $A_* = I_{\mathfrak{S}_{*1}} - \hat{\Theta}_1 D$; by virtue of condition $(\gamma)_0$ we have $DA_* = (I_{\mathfrak{S}_1} - D\hat{\Theta}_1)D = 0$, whence $A_*: \mathfrak{S}_{*1} \to \mathfrak{S}_{*2} (= \mathfrak{D})$. The intertwining properties of $\hat{\Theta}_1$ and D imply that $A_* \in \mathscr{I}(S_{*1}, S_{*2})$. Furthermore, set

$$A'_* = I_{\mathfrak{S}_{*1}}|\mathfrak{D}, \quad B = -\hat{\Delta}_1 D, \quad C = C' = I_{\mathfrak{R}_1}.$$

It is easy to show that all the intertwining properties hold, and so do conditions $(\alpha)_0$—$(\eta')_0$; indeed,

$$(\alpha)_0\colon \quad A_*\hat{\Theta}_1 = (I_{\mathfrak{S}_{*1}} - \hat{\Theta}_1 D)\hat{\Theta}_1 = \hat{\Theta}_1 - (\hat{\Theta}_1 D)\hat{\Theta}_1 = 0, \quad \text{by} \quad (\gamma)_0,$$
$$(\beta)\colon \quad B\hat{\Theta}_1 + C\hat{\Delta}_1 = -\hat{\Delta}_1 D\hat{\Theta}_1 + \hat{\Delta}_1 = 0, \quad \text{by} \quad (\gamma)_0,$$
$$(\gamma_*)_0\colon \quad A'_*A_* + \hat{\Theta}_1 D = (I_{\mathfrak{S}_{*1}} - \hat{\Theta}_1 D) + \hat{\Theta}_1 D = I_{\mathfrak{S}_{*1}},$$
$$(\gamma'_*)_0\colon \quad A_*A'_* = (I_{\mathfrak{S}_{*1}} - \hat{\Theta}_1 D)|\mathfrak{D} = I_{\mathfrak{S}_{*1}}|\mathfrak{D} = I_{\mathfrak{S}_{*2}},$$
$$(\eta')_0\colon \quad DA'_* = D|\mathfrak{D} = 0.$$

So we have proved:

Theorem 1. 4. *The contraction T_1 is similar to some isometry if and only if $\hat{\Theta}_1$ has a left-inverse $D \in \mathscr{I}(S_{*1}, S_1)$. The unitary part of this isometry must be equal to R_1 (up to unitary equivalence).*

Corollary. *T_1 is similar to some unilateral shift if and only if $T_1^{*^n} \to 0$ $(n \to \infty)$ and $\hat{\Theta}_1$ has a left inverse $D \in \mathscr{I}(S_{*1}, S_1)$.*

Proof. Necessity of $T_1^{*^n} \to 0$ follows from the same property of unilateral shifts. On the other hand, this condition is equivalent to $\mathfrak{R}_1 = \{0\}$; *cf.* [5], Chapter II, Theorem 1. 2 and formulas (2. 1), (2. 7). The isometry to which T_1 is similar by virtue of Theorem 1. 4 must therefore have $\mathfrak{R}_2 = \{0\}$, i.e. be a unilateral shift.

7. If T_2 is *unitary*, we not only have $\mathfrak{S}_2 = \{0\}$, but $\mathfrak{S}_{*2} = \{0\}$ as well, so the operators A_*, A'_* are also zero, and the set of conditions $(\alpha)_0$—$(\eta')_0$ reduces to the following:

$$(\beta)_{00} \quad B\hat{\Theta}_1 + C\hat{\Delta}_1 = 0, \quad (\gamma_*)_{00} \quad \hat{\Theta}_1 D = I_{\mathfrak{S}_{*1}}, \quad (\gamma)_{00} \quad D\hat{\Theta}_1 = I_{\mathfrak{S}1},$$

with C boundedly invertible. Thus a necessary condition for T_1 to be similar to some unitary operator is that $\hat{\Theta}_1$ be boundedly invertible. This condition is also sufficient. For, if we choose for T_2 any unitary operator U unitarily equivalent to R_1 and for C any unitary operator satisfying $UC = CR_1$, then the operators

$$D = \hat{\Theta}_1^{-1} \quad \text{and} \quad B = -C\hat{\Delta}_1 \hat{\Theta}_1^{-1}$$

will obviously satisfy the conditions above as well as the intertwining conditions (σ).

So we have

Theorem 1. 5. *The contraction T_1 is similar to some unitary operator if and only if the operator $\hat{\Theta}_1$ is boundedly invertible. This unitary operator must then be equal to R_1 (up to unitary equivalence).*

See [5], Sec. IX. 1 for another proof.

2. Completely non-unitary contractions

1. For c.n.u. contractions we shall use their functional model. All Hilbert spaces to be considered are separable.

For a Hilbert space $\mathfrak{E}$, $L^2(\mathfrak{E})$ will denote the Hilbert space of $\mathfrak{E}$-vector valued functions $u = u(z)$ on the unit circle $(z = e^{it})$, which are (strongly) measurable and norm-square integrable with respect to normed Lebesgue measure, i.e. with

$$\|u\| = \left(\frac{1}{2\pi} \int_0^{2\pi} |u(z)|^2 \, dt \right)^{1/2},$$

where $|\cdot|$ denotes vector norm in $\mathfrak{E}$. Then $H^2(\mathfrak{E})$ is the Hardy subspace of $L^2(\mathfrak{E})$.

We shall be also considering functions $\Phi=\Phi(z)$, whose values are operators from a Hilbert space $\mathfrak{E}$ into a Hilbert space $\mathfrak{F}$; we require that these operator-valued functions be (strongly) measurable and essentially bounded, i.e. with

$$\operatorname{ess\,sup} |\Phi(z)|<\infty;$$

here $|\cdot|$ denotes the norm of operator from $\mathfrak{E}$ into $\mathfrak{F}$. Multiplication on $L^2(\mathfrak{E})$ by such a bounded measurable function Φ is an operator from $L^2(\mathfrak{E})$ into $L^2(\mathfrak{F})$, which we denote by the same letter Φ; thus

$$(\Phi u)(z)=\Phi(z)u(z) \quad (u\in L^2(\mathfrak{E})).$$

Note that the norm $\|\Phi\|$ of this operator equals the essential supremum of $|\Phi(z)|$. In particular, the operator Φ is a contraction if and only if the function Φ is "contractive", i.e. if its values are contractions $\Phi(z):\mathfrak{E}\to\mathfrak{F}$ a.e. on the unit circle.

A bounded measurable function Φ is *analytic* if the corresponding operator Φ maps the subspace $H^2(\mathfrak{E})$ of $L^2(\mathfrak{E})$ into the subspace $H^2(\mathfrak{F})$ of $L^2(\mathfrak{F})$, or equivalently, if its values $\Phi(z)$ are the radial (strong) limits, a.e. on the unit circle, of a bounded holomorphic function $\Phi(\lambda)$ in the open unit disc, $|\lambda|<1$.

Let Θ be a contractive analytic function with values operators $\Theta(z):\mathfrak{E}\to\mathfrak{E}_*$, and which is, moreover, "pure" in the sense that it also satisfies

$$|\Theta(0)a|<|a| \quad \text{for all} \quad a\in\mathfrak{E}, \quad a\neq 0.$$

We associate with Θ the function

$$\Delta(z) = [I_{\mathfrak{E}}-\Theta(z)^*\Theta(z)]^{1/2},$$

which is also measurable and whose values are selfadjoint operators on $\mathfrak{E}$, bounded by 0 and 1. We form the Hilbert space

$$\mathfrak{K} = H^2(\mathfrak{E}_*)\oplus\overline{\Delta L^2(\mathfrak{E})}$$

(where the closure is in the metric of $L^2(\mathfrak{E})$) and its subspace

$$\mathfrak{H} = \left[H^2(\mathfrak{E}_*)\oplus\overline{\Delta L^2(\mathfrak{E})}\right]\ominus\{\Theta u\oplus\Delta u:\ u\in H^2(\mathfrak{E})\},$$

and define on $\mathfrak{H}$ the operator $S(\Theta)$ by

$$S(\Theta)(u\oplus v) = P_{\mathfrak{H}}(\chi u\oplus\chi v),$$

where $\chi(z)\equiv z$ and $P_{\mathfrak{H}}$ denotes orthogonal projection of $\mathfrak{K}$ onto its subspace $\mathfrak{H}$.

This operator $S(\Theta)$ is a c.n.u. contraction, and moreover, one obtains in this way *all* c.n.u. contractions T, up to unitary equivalence. For T given, one has indeed to choose for $\Theta(\lambda)$ the "characteristic" function of T. See [5], Chapter VI.

The operator V defined on the space $\mathfrak{K}$ by

$$V(u\oplus v) = \chi u\oplus\chi v$$

turns out to be the minimal isometric dilation of $T=S(\Theta)$, and in the Wold decomposition of $\mathfrak{K}$ for V we have

$$\mathfrak{S}_*=H^2(\mathfrak{E}_*) \quad \text{and} \quad \mathfrak{R}=\overline{\Delta L^2(\mathfrak{E})}$$

(with the natural embeddings in $\mathfrak{K}$ as $H^2(\mathfrak{E}_*)\oplus\{0\}$ and $\{0\}\oplus\overline{\Delta L^2(\mathfrak{E})}$). The corresponding parts S_* and R of V both are multiplication by χ. On the other hand we have

$$\mathfrak{S} = \mathfrak{K}\ominus\mathfrak{H} = \{\Theta u\oplus\Delta u : u\in H^2(\mathfrak{E})\}.$$

As $u \to \Theta u\oplus\Delta u$ is a unitary map of $H^2(\mathfrak{E})$ onto $\mathfrak{S}$, which commutes with multiplication by χ, it is justified to identify $\mathfrak{S}$ with $H^2(\mathfrak{E})$; S will then be represented by multiplication by χ on $H^2(\mathfrak{E})$. The projection operators $\hat{\Theta}$ and $\hat{\Delta}$, from $\mathfrak{S}$ into $\mathfrak{S}_*$ and $\mathfrak{R}$, will be represented by the restrictions to $H^2(\mathfrak{E})$ of the operators Θ and Δ, respectively.

We shall use the fundamental fact that if $\mathfrak{E}$ and $\mathfrak{E}'$ are Hilbert spaces, and if Ω and Ω' are bounded measurable functions with values operators

$$\Omega(z):\mathfrak{E}\to\mathfrak{E}, \quad \Omega'(z):\mathfrak{E}'\to\mathfrak{E}',$$

then those operators

a) Φ: $H^2(\mathfrak{E})\to H^2(\mathfrak{E}')$,

b) Φ: $H^2(\mathfrak{E})\to\overline{\Omega' L^2(\mathfrak{E}')}$,

c) Φ: $\overline{\Omega L^2(\mathfrak{E})}\to\overline{\Omega' L^2(\mathfrak{E}')}$

which commute with multiplication by χ can be represented as multiplication (on the left) by an operator valued, bounded function $\Phi(\cdot)$ which is

a) analytic, with values $\Phi(z)$: $\mathfrak{E}\to\mathfrak{E}'$ a.e.,

b) measurable, with values $\Phi(z)$: $\mathfrak{E}\to\overline{\Omega'(z)\mathfrak{E}'}$ a.e.,

c) measurable, with values $\Phi(z)$: $\overline{\Omega(z)\mathfrak{E}}\to\overline{\Omega'(z)\mathfrak{E}'}$ a.e.

Here, in case c), "measurability" means that there exists a measurable function Ψ with values $\Psi(z):\mathfrak{E}\to\mathfrak{E}'$ such that

$$\Phi(z) = \Psi(z)|\overline{\Omega(z)\mathfrak{E}} \quad \text{a. e.}$$

For the case a) the above fact is proved e.g. in [5], Sec. V. 3; the cases b) and c) can be dealt with in an analogous manner.

2. Consider now *two* c.n.u. contractions, or rather their functional models, say

$$T_1=S(\Theta_1) \quad \text{and} \quad T_2=S(\Theta_2),$$

where Θ_k are purely contractive analytic functions with values operators $\Theta_k(z):\mathfrak{E}_k \to \mathfrak{E}_{*k}$ $(k=1, 2)$. Then

$$\mathfrak{K}_k = H^2(\mathfrak{E}_{*k}) \oplus \overline{\Delta_k L^2(\mathfrak{E}_k)} \quad (k=1, 2)$$

are the corresponding dilation spaces; the elements of $\mathfrak{K}_k$ can also be thought of as column vectors $\begin{bmatrix} u \\ v \end{bmatrix}$.

For these operators, Lemmas 1. 1 and 1. 2 appear in the following form:

Lemma 2. 1. *The general form of an operator $Y \in \mathcal{I}^+(T_1, T_2)$ is multiplication (on $\mathfrak{K}_1$) by a matrix function*

$$Y(z) = \begin{bmatrix} A_*(z) & 0 \\ B(z) & C(z) \end{bmatrix} \tag{2. 1}$$

where A_* is a bounded analytic function and B, C are bounded measurable functions with values operators

$$A_*(z)\colon \mathfrak{E}_{*1} \to \mathfrak{E}_{*2}, \quad B(z)\colon \mathfrak{E}_{*1} \to \overline{\Delta_2(z)\mathfrak{E}_1}, \quad C(z)\colon \overline{\Delta_1(z)\mathfrak{E}_1} \to \overline{\Delta_2(z)\mathfrak{E}_2}, \tag{2. 2}$$

a.e. on the unit circle, satisfying the conditions

$$A_*\Theta_1 = \Theta_2 A, \quad B\Theta_1 + C\Delta_1 = \Delta_2 A, \tag{2. 3}$$

where A is some bounded analytic function with values operators

$$A(z)\colon \ \mathfrak{E}_1 \to \mathfrak{E}_2 \quad \text{a.e.} \tag{2. 4}$$

Lemma 2. 2. *The general form of an operator $Y \in \mathcal{I}^+(T_1, T_2)$ satisfying $\pi_{12}(Y)=0$ is multiplication (on $\mathfrak{K}_1$) by a matrix function*

$$\begin{bmatrix} \Theta_2(z)D(z) & 0 \\ \Delta_2(z)D(z) & 0 \end{bmatrix},$$

where D is a bounded analytic function, with values operators

$$D(z)\colon \ \mathfrak{E}_{*1} \to \mathfrak{E}_2 \quad \text{a.e.} \tag{2. 5}$$

3. Let us consider besides the functions $\Delta_k(z)$ their duals

$$\Delta_{*k}(z) = [I_{\mathfrak{E}_{*k}} - \Theta_k(z)\Theta_k(z)^*]^{1/2} \quad (k=1, 2).$$

Then $\Theta_k\Delta_k = \Delta_{*k}\Theta_k$. Suppose A_*, A, B, C are functions satisfying the conditions of Lemma 2. 1 and derive from them the function

$$E(z) = [B(z)\Delta_{*1}(z) - C(z)\Theta_1(z)^*] \,|\, \overline{\Delta_{*1}(z)\mathfrak{E}_{*1}}.$$

Clearly, E is a bounded measurable function such that

$$E(z)\colon \overline{\Delta_{*1}(z)\mathfrak{E}_{*1}} \to \overline{\Delta_2(z)\mathfrak{E}_2} \quad \text{a. e.} \tag{2.6}$$

Then, using (2. 3) we get

$$E\Theta_1\Delta_1 = (B\Delta_{*1}\Theta_1 - C\Theta_1^*\Theta_1)\Delta_1 = (B\Theta_1\Delta_1 - C + C\Delta_1^2)\Delta_1 = (\Delta_2 A\Delta_1 - C)\Delta_1$$

and therefore

$$C(z) = [-E(z)\Theta_1(z) + \Delta_2(z)A(z)\Delta_1(z)] \,\big|\, \overline{\Delta_1(z)\mathfrak{E}_1}. \tag{2.7}$$

Furthermore, we have

$$E\Delta_{*1} = B\Delta_{*1}^2 - C\Theta_1^*\Delta_{*1} = B - B\Theta_1\Theta_1^* - C\Delta_1\Theta_1^*;$$

and hence by (2. 3):

$$B = E\Delta_{*1} + \Delta_2 A\Theta_1^*. \tag{2.8}$$

Conversely, for an arbitrary bounded measurable function E with values operators as in (2. 6), the functions B and C generated by (2. 7) and (2. 8) will satisfy conditions (2. 2) and (2. 3). Indeed, we have in particular

$$\begin{aligned} B\Theta_1 + C\Delta_1 &= [E\Delta_{*1} + \Delta_2 A\Theta_1^*]\Theta_1 + [-E\Theta_1 + \Delta_2 A\Delta_1]\Delta_1 = \\ &= E[\Delta_{*1}\Theta_1 - \Theta_1\Delta_1] + \Delta_2 A = \Delta_2 A. \end{aligned}$$

Thus we can give Lemma 2. 1 the following alternative form:

Lemma 2. 1′. *The general form of an operator $Y \in \mathscr{I}^+(T_1, T_2)$ is multiplication by a matrix function* (2. 1), *where A_* is as in Lemma* 2. 1,[1] *while B and C derive by means of formulas* (2. 7) *and* (2. 8) *from some bounded measurable function E with values operators*

$$E(z)\colon \overline{\Delta_{*1}(z)\mathfrak{E}_{*1}} \to \overline{\Delta_2(z)\mathfrak{E}_2}.$$

4. The similarity theorems 1. 3 and 1. 3′ can be formulated for operators $T_k = S(\Theta_k)$ $(k=1, 2)$ as follows:

Theorem 2. 3. *The operators $S(\Theta_k)$ $(k=1, 2)$ are similar if and only if there exist bounded analytic functions A_*, A'_*, A, A', D, D' and bounded measurable functions B, B', C, C' with values operators*

$$(\sigma_1)\begin{cases} A_*(z)\colon \mathfrak{E}_{*1} \to \mathfrak{E}_{*2}, & A(z)\colon \mathfrak{E}_1 \to \mathfrak{E}_2, \quad D(z)\colon \mathfrak{E}_{*1} \to \mathfrak{E}_1, \\ A'_*(z)\colon \mathfrak{E}_{*2} \to \mathfrak{E}_{*1}, & A'(z)\colon \mathfrak{E}_2 \to \mathfrak{E}_1, \quad D'(z)\colon \mathfrak{E}_{*2} \to \mathfrak{E}_2, \end{cases}$$

[1] I. e. bounded analytic with values operators $\mathfrak{E}_{*1} \to \mathfrak{E}_{*2}$, satisfying the relation $A_*\Theta_1 = \Theta_2 A$, where A is some bounded analytic function with values operators $\mathfrak{E}_1 \to \mathfrak{E}_2$.

$$(\sigma_2)\begin{cases} B(z)\colon \mathfrak{E}_{*1}\to\overline{\Delta_2(z)\mathfrak{E}_2}, & C(z)\colon \overline{\Delta_2(z)\mathfrak{E}_2}\to\overline{\Delta_1(z)\mathfrak{E}_1},\\ B'(z)\colon \mathfrak{E}_{*2}\to\overline{\Delta_1(z)\mathfrak{E}_1}, & C'(z)\colon \overline{\Delta_1(z)\mathfrak{E}_1}\to\overline{\Delta_2(z)\mathfrak{E}_2},\end{cases}$$

satisfying a.e. the conditions

(α) $A_*(z)\Theta_1(z)=\Theta_2(z)A(z)$, (β) $B(z)\Theta_1(z)+C(z)\Delta_1(z)=\Delta_2(z)A(z)$,

(α') $A'_*(z)\Theta_2(z)=\Theta_1(z)A'(z)$, (β') $B'(z)\Theta_2(z)+C'(z)\Delta_2(z)=\Delta_1(z)A'(z)$,

(γ_*) $A'_*(z)A_*(z)+\Theta_1(z)D(z)=I_{\mathfrak{E}_{*1}}$, (δ) $B'(z)A_*(z)+C'(z)B(z)=-\Delta_1(z)D(z)$,

(γ'_*) $A_*(z)A'_*(z)+\Theta_2(z)D'(z)=I_{\mathfrak{E}_{*2}}$, (δ') $B(z)A'_*(z)+C(z)B'(z)=-\Delta_2(z)D'(z)$,

(ε) $C'(z)=C(z)^{-1}$.

Theorem 2. 3′. *The operators $S(\Theta_k)$ $(k=1,2)$ are similar if and only if there exist bounded analytic functions A_*, A'_*, A, A', D, D', and bounded measurable functions B, C and C' satisfying conditions (σ), (α), (α'), (β), (γ_*), (γ'_*) of Theorem* 2. 3 *and conditions*

(γ) $A'(z)A(z)+D(z)\Theta_1(z)=I_{\mathfrak{E}_1}$, (η) $A(z)D(z)=D'(z)A_*(z)$,

(γ') $A(z)A'(z)+D'(z)\Theta_2(z)=I_{\mathfrak{E}_2}$, (η') $A'(z)D'(z)=D(z)A'_*(z)$.

Corollary 1. *The equation*

$$\dim\overline{\Delta_1(z)\mathfrak{E}_1}=\dim\overline{\Delta_2(z)\mathfrak{E}_2}\quad\text{a. e.}\tag{2.9}$$

is a necessary condition for $S(\Theta_1)$ and $S(\Theta_2)$ to be similar.

Proof. Immediate from the invertibility of $C(z)$ a.e.

We shall return in Sec. 3 to the question how (2. 9) can replace in some cases the conditions on B, C, C' in Theorem 2. 3′.

Consider now the case that $\Theta_k(z)$ $(k=1,2)$ are *inner* functions (i.e. with values isometries a.e. on the unit circle). Then $\Delta_k(z)=0$ $(k=1,2)$ a.e., and hence, by (σ_2), the values of B, B', C, C' are operators with range $\{0\}$ so that conditions (β), (β'), (δ), (δ'), (ε) of Theorem 2. 3 become trivial. Thus we have:

Corollary 2. *If $\Theta_k(z)$ $(k=1,2)$ are inner functions then conditions (σ_1), (α) (α'), (γ_*), (γ'_*) are necessary and sufficient for $S(\Theta_1)$ and $S(\Theta_2)$ to be similar.*

Taking $D=0$, $D'=0$ we get a sufficient condition:

Corollary 3. *If $\Theta_k(z)$ $(k=1,2)$ are inner functions then for the similarity of $S(\Theta_1)$ and $S(\Theta_2)$ it is sufficient that there exist bounded analytic functions $A_*(z)$, $A(z)$ with bounded inverses $A_*(z)^{-1}$, $A(z)^{-1}$ such that*

$$A_*(z)\Theta_1(z)=\Theta_2(z)A(z)\quad\text{a. e.}$$

This sufficiency condition was obtained in a direct manner in Moore—Nordgren [4].

5. Theorems 1. 4 and 1. 5 can also be given a functional form, and here one need not restrict himself to c.n.u. operators. Indeed every contraction T is the direct sum of a unitary operator and of a c.n.u. contraction T_1. Clearly T is similar to an isometry or to a unitary operator if and only if so does T_1. As T and T_1 have the same characteristic function Θ we deduce from the theorems above:

Theorem 2. 4. *A contraction T is similar to some isometry or to some unitary operator if and only if its characteristic function Θ has a bounded analytic left-inverse, or inverse, respectively.*

(For the unitary case see also [5], Sec. IX. 1.)

3. An equidimensionality criterion for similarity

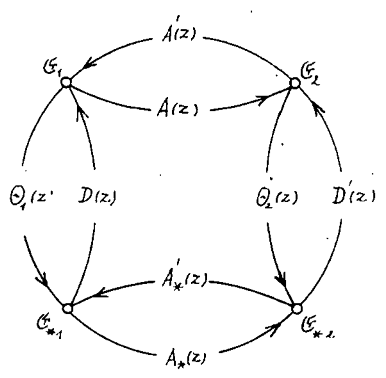
1. The following theorem differs from Theorems 2. 3 and 2. 3′ in that it only involves the analytic functions $A_*, \ldots, D'$, plus the equidimensionality condition (2. 9). More precisely, we prove

Theorem 3. 1. *Suppose $\Theta_k(z)$ $(k=1, 2)$ are purely contractive analytic functions, with values operators $\mathfrak{E}_k \to \mathfrak{E}_{*k}$, and also suppose that the values of the function $\Delta_1(z)$ are compact operators a.e. on the unit circle. Then $S(\Theta_1)$ is similar to $S(\Theta_2)$ if and only if*

$$(*)\qquad \dim \overline{\Delta_1(z)\mathfrak{E}_1} = \dim \overline{\Delta_2(z)\mathfrak{E}_2} \quad a.\ e.$$

and if there exist bounded analytic functions $A_*(z), \ldots, D'(z)$ satisfying conditions (σ_1), (α), (α'), (γ_*), (γ_*'), (γ), (γ'), (η), (η') of Theorems 2. 3 and 2. 3′.

Observe that these conditions can be expressed by the following properties of the diagram: 1) it is commutative along *open* two-step paths (e.g. $AD = D'A_*$); 2) products corresponding to adjacent *closed* two-step paths add to identity (e.g. $A'A + D\Theta_1 = I_{\mathfrak{E}_1}$).



Proof. Necessity follows from Theorem 2. 3′ and its Corollary 1. So we have to prove sufficiency, that is, on account of Theorem 2. 3′, the existence of functions $B(z)$, $C(z)$, $C'(z)$ satisfying conditions (σ_2), (β), (ε). This will be done as follows.

a) First we introduce the spectral family $\{E_x(z)\}_{0\leqq x\leqq 1}$ of the selfadjoint operator $\Delta_1(z)$, normed by the conditions $E_0(z)=0$, $E_1(z)=I$, and, say, continuity from the right for $0<x<1$. As $E_x(z)$ is the limit of a sequence of polynomials of $\Delta_1(z)$, we conclude that $E_x(z)$ is, for any fixed x, a measurable function of $z=e^{it}$ along with $\Theta(z)$ and $\Delta(z)$.

Note that if $a\in[I_{\mathfrak{E}_1}-E_x(z)]\mathfrak{E}_1$ for some x and z $(0\leqq x\leqq 1,\ z=e^{it})$, then

$$(3.1)\qquad |\Delta_1(z)a|\geqq x|a|,\quad |\Theta_1(z)a|^2 = |a|^2-|\Delta_1(z)a|^2 \leqq (1-x^2)|a|^2$$

(norms in $\mathfrak{E}_1$ and $\mathfrak{E}_{*1}$).

Denote by M the least common upper bound of the functions

$$|A_*(z)|,\ \ldots,\ |D'(z)|.$$

For vectors a of the type considered above we have then

$$|a-A'(z)A(z)a| \overset{\gamma}{=} |D(z)\Theta_1(z)a| \leqq M|\Theta_1(z)a| \leqq M(1-x^2)^{1/2}|a|.$$

As we also have

$$|A'(z)A(z)a|\leqq M|A(z)a|$$

we deduce: $|a| \leqq M(1-x^2)^{1/2}|a|+M|A(z)a|$, and hence

$$(3.2)\qquad |A(z)a| \geqq [M^{-1}-(1-x^2)^{1/2}]|a| \geqq \frac{3}{4M}|a|$$

for x close enough to 1. Moreover, we have

$$|\Delta_2(z)A(z)a|^2 = |A(z)a|^2-|\Theta_2(z)A(z)a|^2 =$$

$$\overset{\alpha}{=} |A(z)a|^2-|A_*(z)\Theta_1(z)a|^2 \geqq |A(z)a|^2-M^2|\Theta_1(z)a|^2;$$

using (3. 1) and (3. 2) we get

$$(3.3)\qquad |\Delta_2(z)A(z)a| \geqq \left[\left(\frac{3}{4M}\right)^2-M^2(1-x^2)\right]|a|^2 \geqq \left(\frac{1}{2M}\right)^2|a|^2$$

for x close enough to 1 $(0<x<1)$. Let us fix such a value of x, say ξ, and denote

$$E_\xi^-(z) = I_{\mathfrak{E}_1}-E_\xi(z).$$

On account of (3. 3) we have then for $a\in E_\xi^-(z)\mathfrak{E}_1$

$$(3.4)\qquad |\Delta_2(z)A(z)a| \geqq (2M)^{-1}|a| \geqq (2M)^{-1}|\Delta_1(z)a|.$$

On the other hand, (3. 1) implies, for such a,

$$|\Delta_2(z)A(z)a| \leqq |A(z)a| \leqq M|a| \leqq (M/\xi)|\Delta_1(z)a|. \tag{3.5}$$

b) Now set

$$F(z)=\int_{\xi}^{1}\frac{1}{x}d_x E_x(z) \quad \text{and} \quad G(z)=\Delta_2(z)A(z)F(z); \tag{3.6}$$

these are bounded, measurable functions, and as

$$F(z)a\in E_\xi^-\mathfrak{E}_1 \quad \text{and} \quad \Delta_1(z)F(z)a=a \quad \text{for} \quad a\in E_\xi^-\mathfrak{E}_1 \tag{3.7}$$

we deduce from (3. 4) and (3. 5) that

$$(2M)^{-1}|a|\leqq|G(z)a|\leqq(M/\xi)|a| \quad \text{for} \quad a\in E_\xi^-\mathfrak{E}_1. \tag{3.8}$$

Because $\Delta_1(z)$ is a compact operator for a.e. z, its spectral projection $E_\xi^-(z)$ is of finite rank, a.e. By virtue of (3. 8), $G(z)$ maps $E_\xi^-(z)\mathfrak{E}_1$, for a.e. fixed value of z, bicontinuously onto the space

$$G(z)E_\xi^-(z)\mathfrak{E}_1 \quad (\subset\Delta_2(z)\mathfrak{E}_2),$$

which is therefore of the same dimension as $E^-(z)\mathfrak{E}_1$. From the hypothesis $(*)$ it then follows that the spaces

$$\mathfrak{M}_1(z) = \overline{\Delta_1(z)\mathfrak{E}_1}\ominus E_\xi^-\mathfrak{E}_1 \quad \text{and} \quad \mathfrak{M}_2(z) = \overline{\Delta_2(z)\mathfrak{E}_2}\ominus G(z)E_\xi^-(z)\mathfrak{E}_1$$

are also equidimensional. Thus

$$\dim\mathfrak{M}_1(z)=\dim\mathfrak{M}_2(z)=d(z) \quad \text{a.e.},$$

where $d(z)$ is a measurable function with the possible values $0, 1, \ldots, \infty(=\aleph_0)$. Note that

$$\overline{\Delta_1(z)\mathfrak{E}_1} = (I_{\mathfrak{E}_1}-E_{+0}(z))\mathfrak{E}_1, \quad \text{and hence} \quad \mathfrak{M}_1(z) = (E_\xi(z)-E_{+0}(z))\mathfrak{E}_1.$$

By an appropriate orthogonalization procedure (commonly used in reduction theory) we construct sequences $\{\varphi_k\}_1^\infty$, $\{\psi_k\}_1^\infty$ of $\mathfrak{E}_1$- and $\mathfrak{E}_2$-vector valued measurable functions such that if σ_n denotes, for $n=0, 1, \ldots, \infty$, the set of points z on the unit circle where

$$d(z)=n,$$

then for every fixed $z\in\sigma_n$ the values

$$\varphi_1(z), \ldots, \varphi_n(z) \quad \text{and} \quad \psi_1(z), \ldots, \psi_n(z)$$

form orthonormal bases of $\mathfrak{M}_1(z)$ and $\mathfrak{M}_2(z)$, respectively. Then it is easy to define a measurable function with values *unitary* operators

$$U(z)\colon\ \mathfrak{M}_1(z)\to\mathfrak{M}_2(z),$$

notably we set

$$U(z)\varphi_k(z)=\psi_k(z)\quad\text{for}\quad z\in\sigma_n\quad\text{and}\quad k=1,\dots,n,$$

and extend linearly. (On σ_0 we can set e.g. $U(z)=0$.)

Now consider the function

$$C(x)=\left[G(z)E_\xi^-(z)+U(z)\left(E_\xi(z)-E_{+0}(z)\right)\right]\Big|\overline{\Delta_1(z)\mathfrak{E}_1}\,; \tag{3.9}$$

this is also measurable and its values are operators

$$C(z)\colon\ \overline{\Delta_1(z)\mathfrak{E}_1}\to\overline{\Delta_1(z)\mathfrak{E}_2}\quad\text{(onto)} \tag{3.10}$$

satisfying, by virtue of (3. 8), the inequalities

$$M_1|a|\leqq|C(z)a|\leqq M_2|a|\quad\text{for}\quad a\in\overline{\Delta_1(z)\mathfrak{E}_1} \tag{3.11}$$

with the constants

$$M_1=\min\{1,(2M)^{-1}\},\quad M_2=\max\{1,M/\xi\}.$$

Hence the function

$$C'(z)=C(z)^{-1}$$

has sense, is measurable and bounded,

$$|C'(z)|\leqq 1/M_1.$$

c) Consider now the function

$$H(z)=\Delta_2(z)A(z)-C(z)\Delta_1(z). \tag{3.12}$$

As

$$C(z)\Delta_1(z)E_\xi^-(z)=C(z)E_\xi^-(z)\Delta_1(z)=G(z)E_\xi^-(z)\Delta_1(z)=$$
$$=\Delta_2(z)A(z)F(z)\Delta_1(z)E_\xi^-(z)=\Delta_2(z)A(z)E_\xi^-(z)$$

by virtue (3. 9), (3. 6) and (3. 7), we have $H(z)E_\xi^-(z)=0$. Therefore and from (3. 9) and (3. 6) we have

$$H(z)=H(z)E_\xi(z)=\Delta_2(z)A(z)E_\xi(z)-U(z)\Delta_1(z)\left(E_\xi(z)-E_{+0}(z)\right).$$

Hence,

$$|H(z)a|\leqq(M+1)|E_\xi(z)a|\quad\text{for}\quad a\in\mathfrak{E}_1.$$

On the other hand,

$$|\Theta_1(z)a|^2 = |a|^2 - |\Delta_1(z)a|^2 = \int_0^1 (1-x^2)\, d_x|E_x(z)a|^2 \geqq$$

$$\geqq (1-\xi^2)|E_\xi(z)a|^2 \quad \text{for} \quad a \in \mathfrak{E}_1;$$

combination of the two results gives

$$|H(z)a| \leqq N \cdot |\Theta_1(z)a| \quad \text{for} \quad a \in \mathfrak{E}_1$$

with the constant $N = (M+1)(1-\xi^2)^{-1/2}$.

This shows that, for a.e. fixed z, the operator $B_0(z)$ defined on $\Theta_1(z)\mathfrak{E}_1$ by

$$B_0(z)\Theta_1(z)a = H(z)a \qquad (a \in \mathfrak{E}_1) \tag{3. 13}$$

is (linear and) bounded by N; its definition extends by continuity to the closure of $\Theta_1(z)\mathfrak{E}_1$. Denote by $P(z)$ the orthogonal projection of $\mathfrak{E}_{*1}$ onto its subspace $\overline{\Theta_1(z)\mathfrak{E}_1}$ and set

$$B(z) = B_0(z)P(z). \tag{3. 14}$$

On account of (3. 12) and (3. 10) the range of $H(z)$ is contained in $\overline{\Delta_2(z)\mathfrak{E}_2}$. From these results we conclude:

$$B(z)\colon \mathfrak{E}_{*1} \to \overline{\Delta_2(z)\mathfrak{E}_2}, \quad |B(z)| \leqq N,$$

and

$$B(z)\Theta_1(z) = B_0(z)P(z)\Theta_1(z) = B_0(z)\Theta_1(z) = H(z), \tag{3. 15}$$

i.e. the function $B(z)$ is bounded and satisfies conditions (σ_2) and (β) of Theorem 2. 3. It remains to prove that it is also measurable.

To this effect first note that, by its definition (3. 12), $H(z)$ is measurable. From (3. 15) it follows therefore that $B(z)u(z)$ is measurable for every function $u \in \Theta_1 L^2(\mathfrak{E}_1)$, and hence for every function $u \in \overline{\Theta_1 L^2(\mathfrak{E}_1)}$ also. Next note that since $\Theta_1(z)$ is a measurable function, so is $P(z)$; and hence $w(z) = P(z)v(z)$ is measurable for every $\mathfrak{E}_1$-vector valued measurable function v, in particular for every function $v \in L^2(\mathfrak{E}_1)$. Now in this case it is obvious that w is the orthogonal projection of v onto the subspace $\overline{\Theta_1 L^2(\mathfrak{E}_1)}$ of $L^2(\mathfrak{E}_1)$. Since we have, moreover,

$$B(z)v(z) = B(z)w(z)$$

on account of (3. 14), we conclude that $B(z)v(z)$ is measurable for every $v \in L^2(\mathfrak{E}_1)$, thus $B(z)$ itself is measurable.

This completes the proof of Theorem 3. 1. Observe that the compacity assumption on $\Delta_1(z)$ can be weakened: all we have used is that, for a certain $\xi < 1$, the spectral projection $E_\xi^-(z)$ is of finite rank, a.e.

2. If both $\Theta_1(z)$ and $\Theta_2(z)$ are *scalar* valued then so are all functions occurring in Theorems 2. 3, 2. 3′, and 3. 1, and therefore commute. From conditions (α), (γ) it follows in this case

$$A_*\Theta_1 A' = \Theta_2 AA' = \Theta_2 A'A = \Theta_2(I-D\Theta_1), \quad \Theta_2 = (A_*A'+\Theta_2 D)\Theta_1;$$

and from (α) and (γ_*') we get on a similar way

$$\Theta_1 = (A_*'A+D'\Theta_1)\Theta_2.$$

Thus both functions

(3. 16) $\qquad \Theta_1(z)/\Theta_2(z)$ and $\Theta_2(z)/\Theta_1(z)$ belong to H^∞.

Conversely, if (3. 16) holds then conditions (α)—(η') are fulfilled e.g. by the functions

$$A_*=\Theta_2/\Theta_1,\ A=1,\ D=0;\quad A_*'=\Theta_1/\Theta_2,\ A'=1,\ D'=0.$$

Clearly, $\dim \Delta_k(z)\mathfrak{E}_k$ is 0 or 1 according as $|\Theta_k(z)|$ is <1 or $=1$. Thus we obtain from Theorem 3. 1:

Corollary 1. *Let $\Theta_1(z)$, $\Theta_2(z)$ be scalar valued contractive analytic functions. Then the corresponding operators $S(\Theta_1)$, $S(\Theta_2)$ are similar if and only if*

(i) *$\Theta_1(z)/\Theta_2(z)$ and $\Theta_2(z)/\Theta_1(z)$ belong to H^∞,*

(ii) *the sets $\{z:|\Theta_k(z)|=1\}$ $(k=1, 2)$ coincide up to subsets of zero measure.*

This result was obtained earlier by KRIETE [3].

3. We shall now consider an $N\times N$ matrix valued, purely contractive analytic function

$$\Theta(z)=[\vartheta_{ik}(z)] \qquad (i, k=1, 2, \dots, N).$$

Let

$$\Omega(z)=[\omega_{ik}(z)] \qquad (i, k=1, 2, \dots, N)$$

denote the algebraic adjoint matrix; then

(3. 17) $\qquad \Omega(z)\Theta(z)=\Theta(z)\Omega(z)=d(z)I_N,$ where $d(z)=\det\Theta(z)$.

If Θ is *inner*, it is known that the operator $S(\Theta)$ is *quasisimilar* to an operator $S(\vartheta)$ generated by a scalar valued inner function $\vartheta(z)$ if and only if the functions $\omega_{ik}(z)$ have no non-constant inner common divisor, and in this case we necessarily have $\vartheta=d$. (See [3], Sec. IX. 2, and [6].)

Still in the case of inner Θ, a necessary condition for $S(\Theta)$ to be *similar* to $S(d)$ was proved in [8], Sec. 8 (see in particular the last row on p. 17). In an equivalent form, this condition reads as follows:

$$\text{(c)}\quad \begin{cases} \text{There exist functions } u_i, v_i, w \in H^\infty \quad (i=1, \dots, N) \text{ such that} \\ \sum\limits_{i,j=1}^{N} u_i \omega_{ij} v_j + dw = 1. \end{cases}$$

Necessity of condition (c) easily follows, even for not necessarily inner Θ, from Theorem 2. 3 when applied to $\Theta_1 = \Theta$ and $\Theta_2 = d$. Indeed, conditions (α), (γ_*') and equation (3. 17) give:

$$A_* \Theta = dA, \quad A_* A_*' + dD = 1, \quad \Omega\Theta = \Theta\Omega = d \cdot I_N;$$

hence

$$(dA)\Omega A_*' = (A_* \Theta)\Omega A_*' = A_* dA_*' = d(1 - dD')$$

and dividing by d,

$$A\Omega A_*' = 1 - dD'. \tag{3.18}$$

Since by condition (σ_1) the values of the functions A, A_*', D' have to be operators $E^1 \to E^N$, $E^N \to E^1$, $E^1 \to E^1$, respectively, i.e. of the "matrix" form

$$A = [u_1, \dots, u_N], \quad A_*' = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad D' = [w] \quad \text{with} \quad u_i, v_i, w \in H^\infty, \tag{3.19}$$

(c) immediately follows from (3. 19).

In [6] we did not ask whether for an inner Θ condition (c) is also sufficient for $S(\Theta)$ to be similar to $S(d)$ Now we shall show that it *is* sufficient, even for not inner Θ, if we add the condition

$$\dim \Delta(z) E^N \leqq 1$$

which in our case is equivalent to ($*$).

Thus suppose (c) holds and write it in the form of a congruence modulo d in the algebra H^∞:

$$\sum_{i,j} u_i \omega_{ij} v_j \equiv 1 \pmod{d}. \tag{3.20}$$

By virtue of a well-known theorem in matrix theory we have

$$\omega_{ij}\omega_{kh} - \omega_{ih}\omega_{kj} = \pm d \cdot \det [\vartheta_{mn}]_{m \neq i,k;\, n \neq j,h}. \text{ *)} \tag{3.21}$$

From (3. 20) and (3. 21),

$$\omega_{kh} = \sum_{i,j} u_i \omega_{ij} \omega_{kh} v_j \equiv \sum_{i,j} u_i \omega_{ih} \omega_{kj} v_j \pmod{d}. \tag{3.22}$$

*) See e. g. F. R. GANTMACHER, *Matrizenrechnung*. I (Berlin, 1958), p. 20, formula (33).

Thus there exist functions $t_{kh} \in H^\infty$ such that

$$\omega_{kh} = \sum_{i,j} \omega_{kj} v_j u_i \omega_{ih} + d t_{kh} \qquad (k, h = 1, \dots, N). \tag{3.23}$$

Let us set

$$A = [u_1, \dots, u_N], \quad A'_* = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad D = \begin{bmatrix} t_{11} \dots t_{1N} \\ \vdots \qquad \vdots \\ t_{N1} \dots t_{NN} \end{bmatrix}, \quad D' = [w]$$

and

$$A_* = A\Omega, \quad A' = \Omega A'_*.$$

On account of (3. 17) we have then

$$A_* \Theta = A\Omega\Theta = Ad = dA, \quad \Theta A' = \Theta\Omega A'_* = dA'_* = A'_* d,$$

i.e. conditions (α), (α') of Theorem 3. 1. On the other hand, condition (c) implies

$$\left.\begin{matrix} A_* A'_* + dD' \\ AA' + D'd \end{matrix}\right\} = A\Omega A'_* + dw = 1, \tag{3.24}$$

i.e. conditions (γ'_*) and (γ') of Theorem 3. 1.

Next observe that equation (3. 23) takes the forms

$$\Omega = \Omega A'_* A\Omega + dD = \begin{cases} \Omega A'_* A_* + dD, \\ A'A\Omega + dD; \end{cases} \tag{3.25}$$

multipling by Θ on the left or on the right and dividing by d we get

$$I_N = A'_* A_* + \Theta D \quad \text{and} \quad I_N = A'A + D\Theta,$$

i.e. conditions (γ_*) and (γ) of Theorem 3. 1. From (3. 25) we also derive, multiplying by A on the left or by A'_* on the right:

$$A\Omega = AA'A\Omega + AdD \quad \text{and} \quad \Omega A'_* = \Omega A'_* A_* A'_* + dDA'_*,$$

i.e.

$$(I - AA')A_* = dAD \quad \text{and} \quad A'(I - A_* A'_*) = dDA'_*.$$

Recalling (3. 24) and dividing by d we deduce from these equations that

$$D'A_* = AD \quad \text{and} \quad A'D' = DA'_*,$$

i.e. conditions (η) and (η') of Theorem 3. 1.

Therefore Theorem 3. 1 has the following

Corollary 2. *Let $\Theta(z)$ be an $N \times N$ matrix valued, purely contractive analytic function and suppose that* $\det \Theta(z) \not\equiv 0$. *Then condition* (c) *together with condition* $\dim \Delta(z) E^N \leqq 1$ *a.e., are necessary and sufficient for $S(\Theta)$ to be similar to* $S(\det \Theta)$.

4. Theorems on commutants

1. As an instructive application of Lemma 2. 1′ we are going to prove:

Theorem 4. 1. *Let T be a c.n.u. contraction on a separable Hilbert space, whose commutant $(T)'$ consists of operators $\varphi(T)$, where φ is a meromorphic function in the unit circle, of class N_T* (*cf.* [5], Chapter IV). *Then the characteristic function of T has only values isometries and coisometries, a.e. on the unit circle.*

Proof. It suffices to consider an operator $T=S(\Theta)$ generated by a purely contractive analytic function $\Theta(z)$, with values operators $\mathfrak{E}\to\mathfrak{E}_*$. Suppose that the set of points on the unit circle, where both $\Delta(z)$ and $\Delta_*(z)$ are non-zero, is of positive measure. As $\mathfrak{E}$ and $\mathfrak{E}_*$ are separable, this implies that there exist vectors $e\in\mathfrak{E}$ and $e_*\in\mathfrak{E}_*$ such that both $\Delta(z)e$ and $\Delta_*(z)e_*$ are non-zero on some set ω of positive measure. Consider the function E_0, with values operators $\mathfrak{E}_*\to\mathfrak{E}$, defined by

$$E_0(z)a=(a, \Delta_*(z)e_*)\Delta(z)e \quad \text{for} \quad a\in\mathfrak{E}_*.$$

It is bounded, measurable, and so is its restriction

$$E(z)=E_0(z)|\overline{\Delta_*(z)\mathfrak{E}_*},$$

which has values

$$E(z)\colon \overline{\Delta_*(z)\mathfrak{E}_*}\to\overline{\Delta(z)\mathfrak{E}}. \tag{4. 1}$$

Note that

$$(\Delta(z)E(z)\Delta_*(z)e_*, e)=|\Delta_*(z)e_*|^2|\Delta(z)e|^2\neq 0 \text{ on } \omega. \tag{4. 2}$$

Split ω into two disjoint sets of positive measure, say ω_1 and ω_2, and set

$$E_k(z)=i_k(z)E(z) \qquad (k=1,2), \tag{4. 3}$$

where $i_k(z)$ designates the indicator function of the set ω_k on the unit circle. The functions $E_k(z)$ also satisfy (4. 1) so we can apply Lemma 2. 1′ with $\Theta_1=\Theta_2=\Theta$, $\Delta_1=\Delta_2=\Delta$, $\Delta_{*1}=\Delta_{*2}=\Delta_*$, $A_*=0$, $A=0$, and E_k $(k=1, 2)$. Thus we obtain that the operators Y_k of multiplication by the functions

$$Y_k(z)=\begin{bmatrix}0 & 0\\ E_k(z)\Delta_*(z) & -E_k(z)\Theta(z)|\overline{\Delta(z)\mathfrak{E}}\end{bmatrix} \quad (k=1, 2)$$

belong to $\mathscr{I}^+(T, T)$. Let X_k denote the corresponding operators in $\mathscr{I}(T, T)$, i.e. in $(T)'$. We claim that $X_k\neq 0$. For if, e.g., $X_1=0$ then there exists, by Lemma 2. 2, a bounded analytic function $D_1(z)$ such that

$$\Theta(z)D_1(z)=0 \quad \text{and} \quad \Delta(z)D_1(z)=E_1(z)\Delta_*(z)$$

Hence, $D_1(z)=\Delta(z)E_1(z)\Delta_*(z)$ and, by (4. 2),

$$\big(D_1(z)e_*, e\big)=i_1(z)\big(\Delta(z)E(z)\Delta_*(z)e_*, e\big)\begin{cases}\neq 0 & \text{on} \quad \omega_1,\\ =0 & \text{on} \quad \omega_2.\end{cases}$$

But this is impossible on account of the analyticity of the function $\big(D_1(z)e_*, e\big)$.

By the hypothesis of the theorem there exist functions $u_k,\ v_k\in H^\infty$ $(k=1, 2)$ such that $v_k(T)$ is injective and

(4. 4) $$v_k(T)X_k-u_k(T) = 0.$$

As $X_k\neq 0$ we have $u_k(T)\neq 0$. Thus both $u_k(T)$ and $v_k(T)$ are non-zero, and therefore $u_k\neq 0$, $v_k\neq 0$.

Again using Lemma 2. 2 we infer from (4. 4) that there exist analytic functions $D_k(z)$ not necessarily the same as in the above argument such that

$$v_k(z)\begin{bmatrix}0 & 0\\ E_k(z)\Delta_*(z) & -E_k(z)\Theta(z)|_z\end{bmatrix}-u_k(z)\begin{bmatrix}I_{\mathfrak{E}} & 0\\ 0 & I_{\mathfrak{E}_*}|_z\end{bmatrix}=\begin{bmatrix}\Theta(z)D_k(z) & 0\\ \Delta(z)D_k(z) & 0\end{bmatrix}$$

where $|_z$ denotes restriction to $\Delta(z)\mathfrak{E}$. Hence,

(i) $-u_k(z)I_{\mathfrak{E}_*}=\Theta(z)D_k(z)$,

(ii) $v_k(z)E_k(z)\Delta_*(z)=\Delta(z)D_k(z)$,

(iii) $\big(v_k(z)E_k(z)\Theta(z)+u_k(z)I_{\mathfrak{E}}\big)|_z = 0$.

From (i) we infer that $\Theta(z)D_k(z)$ commutes with every operator on $\mathfrak{E}_*$, in particular with $\Delta_*(z)$. As $\Theta(z)\Delta(z)=\Delta_*(z)\Theta(z)$ we deduce using (ii) that

$$\Theta D_k\Delta_*=\Delta_*\Theta D_k=\Theta\Delta D_k=v_k\Theta E_k\Delta_*,$$

and therefore

(4. 5) $$\Theta F_k=0 \quad \text{for} \quad F_k = D_k\Delta_*-v_kE_k\Delta_*.$$

On the other hand, (iii) and (4. 5) imply

$$u_k\Delta F_k = -v_kE_k\Theta\Delta F_k = -v_kE_k\Delta_*\Theta F_k = 0.$$

Since $u(z)\neq 0$ a.e., it follows

(4. 6) $$\Delta F_k=0.$$

Now (4. 5) and (4. 6) imply $F_k=0$, i.e. we have

(4. 7) $$D_k\Delta_*=v_kE_k\Delta_*;$$

again using (ii) we get

$$D_k \Delta_* = \Delta D_k \qquad (k=1, 2). \tag{4. 8}$$

Setting

$$G = u_1 D_2 - u_2 D_1 \tag{4. 9}$$

we have from (i): $\Theta G=0$ while from (4. 8): $G\Delta_* = \Delta G$. Hence, $G\Delta_*^2 = \Delta^2 G$, $G\Theta\Theta^* = \Theta^*\Theta G = 0$, $G\Theta\Theta^* G^* = 0$, $\Theta^* G^* = 0$, $G\Theta = (\Theta^* G^*)^* = 0$, $G\Theta D_k = 0$, and by (i), $u_k G = 0$. Since $u_k(z) \neq 0$ a.e., we conclude: $G=0$. Then, using (4. 7) and (4. 9),

$$(u_1 v_2 E_2 - u_2 v_1 E_1)\Delta_* = (u_1 D_2 - u_2 D_1)\Delta_* = G\Delta_* = 0,$$

and hence

$$(u_1 v_2 i_2 - u_2 v_1 i_1)E = 0.$$

As $E(z)$ is non-zero on ω, its factor must be zero there. But this factor equals $u_1(z)v_2(z)$ on ω_2, and we arrive at a contradiction to the fact that $u_1 v_2 \in H^\infty$ and $u_1 v_2 \neq 0$.

This contradiction proves the theorem.

Corollary. *Let T be as in Theorem* 4. 1 *and suppose, moreover, that its characteristic function $\Theta(z)$ has a scalar multiple $\delta \in H^\infty$, $\delta \neq 0$. Then T belongs to the class C_0; indeed, $\delta(T)=0$.*

Proof. Since the function Θ has a scalar multiple, its values $\Theta(z)$ are boundedly invertible a.e. As an isometry or a coisometry is not invertible unless it is unitary we infer that $\Theta(z)$ is unitary a.e., and as a consequence $T \in C_{00}$ (i.e., $T^n \to 0$, $T^{*n} \to 0$). By [5], Theorem VI. 5. 1, we have then $\delta(T)=0$.

2. Consider the c.n.u. contraction $T=S(\Theta)$ associated with a *scalar* valued purely contractive analytic function $\Theta(z)$ (i.e., $|\Theta(z)| \leqq 1$ and $\Theta(z)$ is not a constant of modulus 1). We shall show that if $\Theta(z) \not\equiv 0$ then $(T)'$ (i.e. $\mathscr{I}(T, T)$) is commutative. We shall even show that any two operators $Y_1, Y_2 \in \mathscr{I}^+(T, T)$ commute.

Let

$$\begin{bmatrix} A_{*k}(z) & 0 \\ B_k(z) & C_k(z) \end{bmatrix} \quad (k=1, 2)$$

be the corresponding matrix functions. As the entries are scalar valued functions, commutativity of Y_1 and Y_2 will be proved if we show that the function

$$F_{12}(z) = B_1(z)A_{*2}(z) + C_1(z)B_2(z)$$

is symmetric in the subscripts 1, 2. Since the values of $B_k(z)$ outside the set $\sigma = \{z : \Delta(z) \neq 0\}$ vanish (*cf.* condition (2. 2) in Lemma 2. 1) it suffices to consider

$F_{12}(z)$ on the set σ. Now by virtue of condition (2. 3) in Lemma 2. 1, we have

$$A_{*k}\Theta=\Theta A_k \quad \text{and} \quad B_k\Theta+C_k\Delta = \Delta A_k \qquad (k=1, 2).$$

Since $\Theta(z)$ cannot vanish on a set of positive measure we deduce from the first equation that $A_{*k}=A_k$, and from the second, that

$$F_{12}(z) = B_1(z)[B_2(z)\Theta(z)+C_2(z)\Delta(z)]/\Delta(z)+C_1(z)B_2(z) \quad \text{on } \sigma,$$

i.e.

$$F_{12}(z) = B_1(z)B_2(z)\Theta(z)/\Delta(z)+B_1(z)C_2(z)+C_1(z)B_2(z) \quad \text{on } \sigma,$$

and the symmetry in the subscripts 1, 2 is apparent.

The case $\Theta(z)\equiv 0$ is different. Consider in this case e.g. the matrices

$$Y_1(z)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad \text{and} \quad Y_2(z)=\begin{bmatrix}0 & 0\\ B(z) & 0\end{bmatrix},$$

where $B(z)$ is any scalar valued, *non-analytic* bounded measurable function. Both matrices satisfy conditions of Lemma 2. 1 (for $\Theta_1=\Theta_2=\Theta=0$ and $A\equiv 0$), thus the corresponding operators Y_1, Y_2 belong to $\mathcal{I}^+(T, T)$. Then $X_1=\pi(Y_1)$ and $X_2=\pi(Y_2)$ belong to $(T)'$. By virtue of the Multiplication Property of the map π given in Sec. 1, we have $X_1X_2-X_2X_1 = \pi(Y_1Y_2-Y_2Y_1)$. Now the operator $Q = Y_1Y_2-Y_2Y_1$ is multiplication by the matrix function

$$Q(z) = \begin{bmatrix}0 & 0\\ -B(z) & 0\end{bmatrix},$$

and this is certainly not of the form

$$\begin{bmatrix}\Theta(z)D(z) & 0\\ \Delta(z)D(z) & 0\end{bmatrix}, \quad \text{i. e.} \quad \begin{bmatrix}0 & 0\\ D(z) & 0\end{bmatrix}$$

with analytic $D(z)$, and therefore, on account of Lemma 2. 2, $\pi(Q)\neq 0$. Thus X_1 and X_2 do not commute.

Observe that the characteristic function $\Theta(z)\equiv 0$ corresponds to an operator of the form

$$T = S\oplus S^*,$$

where S is a simple unilateral shift. That for such a T the commutant is not commutative can also be deduced from the fact proved in [7], Proposition 5, that there exists a non-zero operator X (indeed, a quasi-affinity) such that $S^*X=XS$.

So we have proved:

Theorem 4. 2. *Every c.n.u. contraction T with defect indices 1, 1 has a commutative commutant $(T)'$, with the only exception of the operator $T = S\oplus S^*$, where S is a simple unilateral shift.*

5. Inverse of a function of T

1. Let T be a c.n.u. contraction on the space $\mathfrak{H}$ and let V be its minimal isometric dilation on $\mathfrak{K}$ (we use the notations of Sec. 1). By the functional calculus developed in [5], Chapter III, the operators $u(T)$ and $u(V)$ have sense for every function $u\in H^\infty$ and are connected by the relation

$$u(T)=P_{\mathfrak{H}}u(V)|\mathfrak{H}. \tag{5. 1}$$

If R is the unitary part of V then $u(R)$ also has sense (it is the restriction of $u(V)$ to $\mathfrak{R}$).

Theorem 5. 1. *If $u(T)$ is boundedly invertible then so is $u(R)$ and we have*

$$\|u(R)^{-1}\|\leqq\|u(T)^{-1}\|. \tag{5. 2}$$

Proof. We use the fact, proved in the proof of Proposition II. 6. 2 in [5], that for every $k\in\mathfrak{R}$ there exists a sequence of elements $h_n\in\mathfrak{H}$ such that

$$k=\lim_{n\to\infty} V^n h_n.$$

This implies:

$$\|u(R)k\|=\|u(V)k\|=\lim\|u(V)V^n h_n\|=\lim\|V^n u(V)h_n\|=$$

$$=\lim\|u(V)h_n\|\geqq\lim\inf\|u(T)h_n\|\geqq c\lim\inf\|h_n\|=c\|k\|,$$

where $c=\|u(T)^{-1}\|^{-1}$. As $u(R)$ is normal we conclude that $u(R)$ is boundedly invertible and (5. 2) holds.

2. Thus the existence of $u(R)^{-1}$ is necessary for the existence of $u(T)^{-1}$. Necessary and sufficient conditions follow from results of Sec. 1 when we observe that (1. 3) implies $u(T)P_{\mathfrak{H}}=P_{\mathfrak{H}}u(V)$ so that on account of (1. 5) we have $u(V)\in\in\mathscr{I}^+(T,T)$ and $\pi(u(V))=u(T)$. Using matrices corresponding to the decomposition $\mathfrak{K}=\mathfrak{S}_*\oplus\mathfrak{R}$ we deduce from Lemmas 1. 1, 1. 2, and the Multiplication Property of the map π, that $u(T)$ is boundedly invertible if and only if there exist operators

$A_*\in\mathscr{I}(S_*,S_*)$, $A\in\mathscr{I}(S,S)$, $B\in\mathscr{I}(S_*,R)$, $C\in\mathscr{I}(R,R)$, $D\in\mathscr{I}(S_*,S)$, $D'\in\mathscr{I}(S_*,S)$

satisfying the equations

$(\alpha)\quad A_*\hat\Theta=\hat\Theta A,\qquad(\beta)\quad B\hat\Theta+C\hat\Delta=\hat\Delta A,$

$$(\varkappa)\begin{cases}\begin{bmatrix}A_* & 0\\ B & C\end{bmatrix}\begin{bmatrix}u(S_*) & 0\\ 0 & u(R)\end{bmatrix}=I-\begin{bmatrix}\hat\Theta D & 0\\ \hat\Delta D & 0\end{bmatrix},\\[2ex] \begin{bmatrix}u(S_*) & 0\\ 0 & u(R)\end{bmatrix}\begin{bmatrix}A_* & 0\\ B & C\end{bmatrix}=I-\begin{bmatrix}\hat\Theta D' & 0\\ \hat\Delta D' & 0\end{bmatrix}.\end{cases}$$

As a consequence of the intertwining properties of A_*, B and C we can take $D'=D$ and condition $(\varkappa)$ is equivalent to the following system of conditions:

$$(\varkappa_1)\quad A_* u(S_*)+\hat{\Theta}D = I, \qquad (\varkappa_2)\quad B u(S_*)+\hat{\Delta}D=0, \qquad (\varkappa_3)\quad C=u(R)^{-1}.$$

Taking into account that (1. 14) implies

$$u(S_*)\hat{\Theta}=\hat{\Theta}\,u(S) \quad\text{and}\quad u(R)\hat{\Delta}=\hat{\Delta}\,u(S)$$

we deduce:

$$\hat{\Theta}(A\,u(S)+D\hat{\Theta}-1)\stackrel{\alpha}{=}(A_*u(S_*)+\hat{\Theta}D-I)\hat{\Theta}\stackrel{\varkappa_1}{=}0,$$

$$\hat{\Delta}(A\,u(S)+D\hat{\Theta}-I)\stackrel{\varkappa_{2,3}}{=}\hat{\Delta}A\,u(S)-B\,u(S_*)\hat{\Theta}-C\,u(R)\hat{\Delta}=(\hat{\Delta}A-B\hat{\Theta}-C\hat{\Delta})u(S)\stackrel{\beta}{=}0,$$

and hence

$$(\lambda)\qquad A\,u(S)+D\hat{\Theta} = I.$$

Conversely, (β) and $(\varkappa_2)$ are implied by (λ) and the rest of the conditions if we set

$$B=-C\hat{\Delta}D.$$

Indeed, the intertwining property for B follows immediately from those for C, $\hat{\Delta}$, and D, while $(\varkappa_2)$ follows from the equations

$$B\,u(S_*) = -C\hat{\Delta}D\,u(S) = -C\hat{\Delta}\,u(S)D = -C\,u(R)\hat{\Delta}D = -\hat{\Delta}D;$$

finally, (β) follows from the equations

$$u(R)(B\hat{\Theta}+C\hat{\Delta}-\hat{\Delta}A) = -\hat{\Delta}D\hat{\Theta}+\hat{\Delta}-\hat{\Delta}\,u(S)A = \hat{\Delta}(-D\hat{\Theta}+I-u(S)A)\stackrel{\lambda}{=}0$$

when we multiply by C on the left.

Thus the initial set of conditions can be replaced by the set (α), $(\varkappa_1)$, $(\varkappa_3)$, (λ). Multiplying $(\varkappa_1)$ and (λ) by $\hat{\Theta}$ on the right and on the left, respectively, and using the intertwining properties and substracting we obtain that

$$u(S_*)(A_*\hat{\Theta}-\hat{\Theta}A)=0.$$

As the unilateral shift S_* is the restriction of a bilateral shift U, and hence $u(S_*)$ is a restriction of $u(U)$, and as $u(U)$ has zero null-space for $u\neq 0$ (because then $u(z)\neq 0$ a.e.), we conclude that (α) also holds, i.e. it is a consequence of $(\varkappa_1)$ and (λ).

So we have proved:

Theorem 5. 2. *Let $u\in H^\infty$, $u\neq 0$. In order that $u(T)$ be boundedly invertible it is necessary and sufficient that*

a) *$u(R)$ be boundedly invertible,*
b) *there exist operators $A_*\in\mathscr{I}(S_*, S_*)$, $A\in\mathscr{I}(S, S)$, $D\in\mathscr{I}(S_*, S)$ such that*

$$A\,u(S)+D\hat{\Theta} = I_{\mathfrak{S}}, \quad A_*u(S_*)+\hat{\Theta}D = I_{\mathfrak{S}_*}.$$

Remark. Since $u(R)$ is normal, condition a) is equivalent to the condition that

$$\|u(R)f\| \geqq m\|f\| \quad \text{for some} \quad m>0 \quad \text{and all} \quad f \in \mathfrak{R}.$$

3. If $T=S(\Theta)$, Θ being a purely contractive analytic function with values operators $\Theta(z):\mathfrak{E}\to\mathfrak{E}_*$, then the above conditions a), b) can be expressed in the following form:

a) $|u(z)|\geqq m>0$ at a.e. point z where $\Delta(z)\neq 0$, i.e. $\Theta(z)$ is not an isometry,

b) there exist bounded analytic functions A_*, A, D with values operators

$$A_*(z)\colon \mathfrak{E}_*\to\mathfrak{E}_*, \quad A(z)\colon \mathfrak{E}\to\mathfrak{E}, \quad D(z)\colon \mathfrak{E}_*\to\mathfrak{E} \quad \text{a.e.}$$

such that

$$u(z)A(z)+D(z)\Theta(z) = I_{\mathfrak{E}}, \quad u(z)A_*(z)+\Theta(z)D(z) = I_{\mathfrak{E}_*} \quad \text{a.e.} \tag{5.3}$$

Now $u(T)$ is boundedly invertible if and only if so is $u(T)^*$; and $u(T)^*$ is unitarily equivalent to $u^{\tilde{}}(T')$, where $T'=S(\Theta^{\tilde{}})$. Here we use the notations $u^{\tilde{}}$ and $\Theta^{\tilde{}}$ for the functions defined by

$$u^{\tilde{}}(z)=\overline{u(\bar{z})}, \quad \tilde{\Theta}(z)=\Theta(\bar{z})^*$$

(*cf.* [5], Theorem III. 2. 1 and Chapter VI).

Thus conditions a), b) imply that $|u^{\tilde{}}(z)|\geqq m_*>0$ at a.e. point z where $\Theta^{\tilde{}}(z)$ is not an isometry, i.e. $|u(z)|\geqq m_*$ at a.e. point z where $\Theta(z)$ is not a coisometry. Hence, a), b) imply that $|u(z)|\geqq p(>0)$ at a.e. point z where $\Theta(z)$ is not unitary.

So we have:

Theorem 5. 3. *Let $T=S(\Theta)$ and $u\in H^\infty$, $u\neq 0$. In order that $u(T)$ be boundedly invertible it is necessary and sufficient that there exist bounded analytic functions A_*, A, D satisfying conditions* (5. 3), *and a positive number p such that*

$$|u(z)|\geqq p \quad \textit{at a.e. point} \quad z=e^{it} \quad \textit{where} \quad \Theta(z) \quad \textit{is not unitary.} \tag{5.4}$$

4. Consider the particular case when $\Theta(z)$ is an $N\times N$ matrix valued function, limit on the unit circle of a (purely contractive, analytic) function $\Theta(\lambda)$ on the open unit disc. Let $d(\lambda)=\det\Theta(\lambda)$.

As a contraction on a finite dimensional euclidean space is unitary if and only if its determinant is of absolute value 1, condition (5. 4) can be expressed in the form

$$|u(z)|\geqq p \text{ at a.e. point } z \text{ where } |d(z)|\neq 1. \tag{5.4'}$$

Next we notice that the equations (5. 3) hold in the unit disc as well. Thus at every point λ where $\Theta(\lambda)$ has a bounded inverse we have

$$\Theta(\lambda)^{-1} = u(\lambda)A(\lambda)\Theta(\lambda)^{-1}+D(\lambda),$$

and hence

$$(5.5)\qquad |\Theta(\lambda)^{-1}| \leqq M(|u(\lambda)|\,|\Theta(\lambda)^{-1}|+1), \quad \text{or} \quad |u(\lambda)|+|\Theta(\lambda)^{-1}|^{-1} \geqq 1/M$$

where M equals the larger one of the values $\|A\|_\infty$ and $\|D\|_\infty$. As for every invertible operator Z on E^N we have $|\det Z|^N \geqq |Z^{-1}|^{-1}$ (*cf.* Lemma 2. 3 in [1]), inequality (5. 5) implies

$$|u(\lambda)|+|d(\lambda)|^{1/N} \geqq 1/M,$$

and hence

$$(5.6)\qquad |u(\lambda)|+|d(\lambda)| \geqq q(>0).$$

If $d(\lambda)\not\equiv 0$ then $\Theta(\lambda)^{-1}$ exists at every point λ of the open unit disc, perhaps with the exception of countably many points, therefore (5. 6) holds then everywhere in the unit disc. By virtue of the "Corona Theorem" condition (5. 6) is equivalent to the existence of functions $a, b \in H^\infty$ such that

$$(5.7)\qquad u(\lambda)a(\lambda)+d(\lambda)b(\lambda) = 1.$$

Conversely, (5. 7) implies equations (5. 3), with $A(\lambda)=A_*(\lambda)=a(\lambda)I_N$, and $D(\lambda)= =b(\lambda)\Omega(\lambda)$, where $\Omega(\lambda)$ designates the algebraic adjoint of the matrix $\Theta(\lambda)$.

We state our result as follows:

Theorem 5. 4. *Let $\Theta(\lambda)$ be a purely contractive analytic $N\times N$ matrix function with $d(\lambda)=\det\Theta(\lambda)\not\equiv 0$, and let $T=S(\Theta)$ and $u\in H^\infty$. The operator $u(T)$ is boundedly invertible if and only if there exist constants $p, q>0$ such that*

α) $|u(z)|\geqq p$ *at a.e. point* $z=e^{it}$ *where* $|d(z)|\neq 1$, *and*

β) $|u(\lambda)|+|d(\lambda)| \geqq q$ *at every point* λ, $|\lambda|<1$.

The particular case of this theorem when $\Theta(\lambda)$ is an *inner* function, was considered in FUHRMANN [1]. Let us add that another generalization of Fuhrmann's result was given in HERRERO [2].

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Bibliographie

J. F. Adams, Algebraic Topology—A Student's Guide (London Mathematical Society Lecture Notes Series 4), VI+300 pages, Cambridge, University Press, 1972.

The author's objective is to provide the student with a reliable orientation on the vast and complex material facing him when he tries to master algebraic topology. The unusually long Introduction is the pith of this quite new type of book. The Introduction surveys those topics in algebraic topology which should be studied after a first course in topology, namely categories and functiors; semi-simplicial complexes; ordinary homology and cohomology; spectral sequences; $H^*(BG)$; Eilenberg—MacLane spaces and the Steenrod algebra; Serre's theory of classes of abelian groups (C-theory); obstruction theory; homotopy theory; fibre bundles and topology of groups; generalized cohomology theories. All these topics are dealt with separately, where first the basic ideas underlying the topic are motivated, and then a sketch is given of what the author considers the ideal way of studying the topic. Some very instructive comments are made on the various treatments of the topic available in the literature, where sources recommended in the first place are indicated as well. The second and more voluminous part of the book is a collection of extracts from famous papers. These generally illustrate in a concise way the basic ideas in connection of the above topics.

The book thus cleverly meets the urgent need of those who intend to study the immense and sometimes inconsistent literature of algebraic topology. The fact that it has been written by an author who has authentically indicated the essential points in an overall picture and the technical advantages in different presentations renders this book a very valuable guide for students of algebraic topology.

J. Szenthe (Szeged)

A. Dold, Lectures on algebraic topology (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 200) XI+377 pages, Berlin—Heidelberg—New York, Springer Verlag, 1972.

This book presenting singular homology and cohomology theory, grew out of the author's lectures on algebraic topology and can be highly recommended as a textbook for a one-year course. Chapter I, rather a summary of prerequisites than a detailed presentation, covers categories, Abelian groups and homotopy. Chapter II treats the homology of complexes, and singular homology is introduced in Chapter III. Applications are considered in Chapter IV, where among other things the degree of a map, homology properties of neighborhood retracts in R^n, the Jordan theorem and domain invariance are treated. Cellular decomposition and cellular homology is considered in Chapter V. Functors and complexes are taken up in Chapter VI covering among other things the universal coefficient formula, the Künneth formula and the Eilenberg—Zilber theorem. Various standard products such as the exterior homology and cohomology products, the Pontrjagin product, the cup-product, the cap-product and some others are introduced in Chapter VII. A very rich material concerning manifolds, including the Poincaré —Lefschetz duality, the Thom isomorphism, the Gysin sequence, intersection of homology classes and other subjects, is presented in Chapter VIII.

As the students of a first course in algebraic topology generally form a rather wide variety the choice and presentation of the matrial for such a course is a perplexing problem. By preferring the singular theory the author seems to have attained an optimum solution. In fact on the one hand an ample algebraic foundation with a great number of motivating examples is provided for those who want to study this subject further; on the other hand for those who are interested only in the applications of algebraic topology the singular theory suits equally the best and the material concerning manifolds covers the most important applications. The presentation also attains an optimum in conciseness and readability.

J. Szenthe (Szeged)

F. F. Bonsall and J. Duncan, Numerical ranges of operators on normed spaces and of elements of normed algebras (London Mathematical Society Lecture Note Series, 2), IV+142 pages, Cambridge, University Press, 1971.

The concept of the numerical range for a linear operator on a normed space was introduced in 1961—62 by Lumer and Bauer in distinct, though related manners both generalizing the classical Hilbert space case. For the elements of a normed algebra the numerical range is defined as that of the corresponding left regular representation operator. The numerical range reflects both the algebraic and the metric structure of the algebra in contrast to the spectrum which depends on the algebra structure only.

The present book is the first self-contained exposition of the subject. The authors use only standard "elementary" material on the theory of linear normed spaces and normed algebras. The power of the numerical range concept is shown e. g. in its application to the problem of metric characterization of C^*-algebras, thus adding a useful tool for the Hilbert space operator theory also.

The book consists of four chapters.

Chapter 1 introduces the numerical range of an element a in a unital normed algebra A:

$V(A, a) = \{f(a): f \in A', f(1)=1=\|f\|\}$, A' the dual space of A. It is shown that $V(A, a)$ is a compact convex set in the scalar field (real or complex) depending only on the two-dimensional space spanned by 1 and a. When A is complex and complete, $V(A, a)$ contains the spectrum of a. The numerical radius $v(a) = \max\{|\lambda|: \lambda \in V(A, a)\}$ is immediately seen to be a pseudo-norm on A. For complex A we have $v(a) \leq \|a\| \leq ev(a)$, implying in particular that 1 is a vertex and a point of local uniform convexity of the unit ball of A. The proof of the power inequality $\|a^n\| \leq n!\,(e/n)^n v(a)^n$ and the introduction of the numerical index for A concludes the chapter.

In Chapter 2 those elements h of a complex unital Banach algebra A play central role which have real numerical range. These elements h are called Hermitian and form a real Banach space $H(A)$ such that $i(hk-kh) \in H(A)$ for all $h, k \in H(A)$. It is proved that $h \in H(A)$ if and only if $\|\exp ith\| = 1$ for any real t. Let $J(A)$ denote the set $\{h+ik: h, k \in H(A)\}$. Then $J(A)$ is a complex Banach space which is not necessarily an algebra. It is a subalgebra in A if and only if $h^2 \in J(A)$ for all $h \in H(A)$. The main part of the chapter is the proof of the Vidav-Palmer theorem asserting that A is isometrically isomorphic with a C^*-algebra (i. e. with a norm-closed selfadjoint subalgebra of all bounded linear operators on a Hilbert space) if and only if $A = J(A)$. Among the many applications of this theorem it is shown that a complex Banach $*$-algebra satisfying condition $\|x^*x\| = \|x\|^2$ (B^*-algebra) or $\|x^*x\| = \|x^*\|\,\|x\|$ is isometrically $*$-isomorphic with a C^*-algebra.

Chapter 3 introduces the spatial numerical range of a bounded linear operator T of a normed space X:

$$V(T) = \{f(Tx): x \in X, \|x\|=1, f \in X', \|f\|=1\},$$

X' being the dual space of X. The numerical range of T as an element of the Banach algebra $B(X)$ of all bounded linear operators on X is the closed convex hull of $V(T)$. The numerical range $W(T)$ corresponding to a semi-inner-product $[\cdot]$ is defined by

$$W(T) = \{[Tx, x]: [x, x] = 1\}.$$

It is shown that $V(T)$ contains $W(T)$ and that $W(T)$ has the same closed convex hull as $V(T)$. The rest of the chapter deals with spectral, geometric and topological properties of $V(T)$. For a complex Banach space X the authors prove that the closure of $V(T)$ contains the spectrum of T and that the closure of $W(T)$ contains the boundary of the spectrum of T. A theorem of Nirschl—Schneider asserting that eigenvalues of T that lie on the boundary of the closed convex hull of $V(T)$ have index (ascent) one is also proved. The chapter also contains the result that $V(T)$ is connected.

Chapter 4 contains miscellaneous results. The discussion of the numerical range in the second dual of a Banach algebra gives by the Vidav—Palmer theorem that the second dual of a complex B^*-algebra is again a B^*-algebra. The normalized linear functionals $\Omega(A)$ on a complex unital Banach algebra A which are dominated by the spectral radius are called spectral states. These states annihilate elements of the form ab-ba, with $a, b \in A$. A spectral state is multiplicative if and only if it induces a strictly irreducible representation. It is shown in particular that if H is an infinite dimensional Hilbert space then there are no spectral states on $B(H)$. For finite dimensional algebras a complete description of the spectral states in terms of normalized traces (on matrix algebras) is given. The final section of the chapter contains 17 remarks and 16 problems on isolated topics of the numerical range theory.

The book is indispensable for every student or researcher of Banach algebras.

Z. Sebestyén (Budapest)

Essays on topology and related topics. Mémoires dédiés à Georges de Rham, publiés sous la direction de André Haefliger et Raghavan Narasimhan, XII+252 pages, Springer Verlag, Berlin—Heidelberg—New York, 1970.

Le premier article de ce volume en hommage est un exposé des travaux de G. de Rham sur les variétés différentiables par Henri Cartan, réunissant un ensemble des résultats aujourd'hui classiques dans leur contexte historique. Tels sont les résultats recouverts par la dénomination collective „le théorème de de Rham" fondamentaux pour l'introduction des groupes de chomologie calculés avec les formes différentielles d'une variété différentiable. Les divers résultats obtenus par la notion féconde de „courant" sont rassemblés en indiquant comment ces distributions permettent d'obtenir une homologie avec les formes différentielles d'une variété différentiable orientée. La théorie des variétés riemanniennes est représenté par les résultats sur les formes harmoniques et sur la réductibilité. Les autres articles du volume sont les suivants: J. Milnor et O. Burlet: Torsion et type simple d'homotopie; M. Atiyah and F. Hirzebruch: Spin-Manifolds and group actions; P. F. Baum and R. Bott: On the zeroes of meromorphic-vector-fields; R. Bott and S. S. Chern: Some formulas related to complex transgression; K. Kodaira: On homotopy $K3$ surfaces; A Borel: Pseudo-concavité et groupes arithmétiques; A. Andreotti and G. Tomassini: Some remarks on pseudoconcave manifolds; J. L. Koszul: Trajectoires convexes de groupes affines unimodulaires; E. Vesentini: Maximum theorem for spectra; N. H. Kuiper and B. Terpstra—Keppler: Differentiable close-embeddings of Banach manifolds; M. W. Hirsch: On invariant subsets of hyperbolic sets; W. Browder and T. Petrie: Semi-free and quasi-free S^1 actions on homotopy spheres; S. P. Novikov: Pontrjagin classes, the fundamental group and some problems of stable algebra; J. Boéchat et A. Haefliger: Plongements différentiables des variétés orientées de dimension 4 dans R^7; C. Weber: Taming complexes in the metastable range; B. Eckmann et S. Maumary: Le groupe des types simples d'homotopie sur an polyèdre; J. Tits: Sur le groupe des automorphismes d'un arbre; M. A. Kervaire: Multiplicateurs de Schur et K-théorie; R. Thom: Topologie et linguistique. Le volume est terminé par une liste des publications scientifiques de G. de Rham.

J. Szenthe (Szeged)

D. T. Finkbeiner II, Elements of Linear Algebra (A Series of Books in Mathematics), XI+268 pages, San Francisco, W. H. Freeman and Company Ltd., 1972.

The book covers the material of a first course in linear albegra for college students. The introductory chapter contains a general survey of linear algebra, the other four chapters provide substantial introductions to linear spaces (mostly real and finite dimensional), linear mappings, and matrix algebra, sysytems of linear equations and determinants, characteristic values and diagonalization problems. All the thirty-seven sections end with numerous exercises and suggestions and answers are provided for almost all exercises at the end of the book. Out of the few unprecise statements in the book we mention that (on p. 132) the author says that the spectral theorem is true for precisely those linear transformations that are linear combinations of projections (he omits the modifier "commuting").

J. Szűcs (Szeged)

F. Gécseg and I. Peák, Algebraic theory of automata (Disquisitiones Mathematicae Hungaricae, vol. 2), XIV+326 pages, Akadémiai Kiadó, Budapest, 1972.

In the last decade, a remarkable number of books, (about fifteen), addressed to give a systematical treatment of the theory of automata, has been published. Even having such a considerable rivalry, it seems to be expectable that the work reviewed now will be ranked among the most valued monographies on this subject.

The authors have delimited with an approvable sense the topics contained in the book. They disregarded a few branches of the theory (experiments with automata, Turing machines) lying far from the basic subject; moreover, the stochastic extensions of the automaton-theoretical investigations (an area which had scarcely been studied before the labour on this book) are likewise missing; these spontaneous restrictions, however, made possible that the volume should contain an approximatively complete, systematical, precise treatment of the researches of algebraic character on the discrete deterministic automata.

The book presupposes only minimal previous knowledge; it reckons, however, on readers having mature mathematical abilities. The presentation is exact and concise. The latest mentioned property seems to follow necessarily from how a rich material is condensed in a volume of modest size.

In Chapter 1 (Concept of automaton) the fundamental concepts being in connection with automata are dealt with, the minimization procedure is also included. Chapter 2 (Analysis and synthesis Algebra of events) contains, among others, Kleene's theorems on the regular expressions the results of Red'ko and Salomaa on the axiomatizability of regular expressions, investigations of Brzozowski and Yoeli on the derivation of events and on some special event types. In Chapter 3 (Some special classes of automata), the notions of commutative, nilpotent, definite, linear and push-down automata are introduced and studied. Chapter 4 (Composition of automata. Automation mappings) is mainly devoted to Gécseg's results on the properties of families of automaton mappings and on the metrical completeness of automaton systems with respect to various composition concepts. In Chapter 5 (Automata and semigroups), the characteristic semigroups, endomorphism semigroups and automorphism groups of automata are studied, including a number of theorems due to Peák. In the Appendix of the book (Structural systems), the basic concepts of the structural automaton theory are introduced, together with the theorem of Post and Jablonskij on the functional completeness of truth functions and Quine's method for determining the minimal representations of these functions by disjunctive normal forms.

The volume terminates with a bibliography consisting of about 250 items.

A. Ádám (Budapest)

George Grätzer, Lattice theory (First concepts and distributive lattices), XV+212 pages, W. H. Freeman and Co., San Franciso, 1971.

The aim of this book, divided into three chapters, is to give a detailed presentation of distributive lattices.

In the first chapter the basic concepts of lattice theory are introduced and free lattices, which take an essential part in the book, are very carefully discussed.

The second chapter starts with a skillful development of various characterizations and representations of distributive and Boolean lattices. It follows a section on congruence relations in distributive lattices. The next one, including some new results of the author, deals with some generation problems of distributive lattices, Boolean lattices and Boolean algebras. (Boolean lattices and Boolean algebras are strictly and consistently distinguished here.) Then it comes Stone's representation theory of distributive lattices by the topological spaces defined on the partially ordered sets of their prime ideals. Next, one can find the results of the author and H. Lakser, on free distributive products. The last section of the chapter deals with the characterizations of mono- and epimorphisms as well of certain projective subclasses and all the injective ones in the categories of the distributive lattices, bounded distributive lattices and Boolean algebras.

The third chapter discusses distributive lattices with pseudocomplementation including Stone lattices. All equational classes of them and the subdirectly irreducible ones are completely described. Moreover, Stone lattices are characterized among the distributive ones and conditions are given for a Stone lattice to be injective. Most of the results discussed in this chapter were obtained by the author, partly with R. Balbes and H. Lakser.

Each section is followed by many exercices, and at the end of each chapter several unsolved problems are mentioned.

The author says in the preface that he was to break with the traditional approach to lattice theory, which proceeds from partially ordered sets to general lattices, semimodular lattices, modular lattices and finally distributive lattices. But he does not realize his decision perfectly because he introduces the lattices as partly ordered sets of special kind. In the rest of the book the central part is taken by the distributive lattices indeed and the more general facts needed are introduced just when and where they are applied the first time. This construction, logical and attractive, evidently has disadvantages, too, appearing for instance in the fact that some of the general theorems are discussed in such a section where the reader would not think to find them.

The author strives successfully after estimating the results and showing the inner connections of the matter. But, in the opinion of the reviewer, the remark on priority after Theorem 6.4 is superfluous. The Further Topics and References at the ends of the chapters cannot provide, of course, a complete survey of lattice theory, but some applications to algebra and geometry would have been worthy enough to be mentioned.

The composition of the book is careful. There is only one fault found by the reviewer: one reads the sentence "if L has 0 and 1" on page 23, while these elements will be defined on page 56. only.

Summarizing, Grätzer's book is a good introduction to some problems concerning the class of distributive lattices and some of its important subclasses.

G. Szász (Budapest)

W. Hahn, Stability of Motion (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, 138) XI+446 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1967. — DM 72, —

In the last two decades the number of publications concerning the stability problems of functional equations (differential-, differential-difference-, and difference-equations) has grown enormously. In general, these papers are connected with practical problems and investigate the qualitative behaviour of the given special systems.

The aim of this monograph is twofold: to summarize from a uniform point of view the results of the stability theory of functional equations as a rigorous mathematical discipline, and to present important and comprehensive applications to the theory.

The first part of the book deals with the stability of solutions of special functional equations drawing the attention to analogies between the various results. In Chapter 5 the general concept of motion is defined; then, by the aid of comparison functions, a classification of stability types is given. There follows a number of stability criteria in a general form based on the direct method of Liapunov. The results are illustrated by examples and applications to ordinary differential-, difference-, and partial differential equations. In the further chapters special questions on motions generated by ordinary differential equations, are discussed.

Altogether it is a characteristic feature of monograph that in the setting of problems, interpretation of results and choosing of terminology, it combines in a lucky manner the classical treatment with a treatment aimed at modern physical and technical applications. It deals with problems and methods concerning the stability of automatic control systems and servomechanisms (e. g. speed control by means of a centrifugal pendulum, regulation of the water level in a container, Popov's criterion and so on).

The monograph is beautifully organized and highly readable. It is not only for specialists but a very valuable text book for beginners too.

L. Hatvani (Szeged)

K. Hinderer, Grundbegriffe der Wahrscheinlichkeitstheorie, VIII+247 Seiten, Berlin—Heidelberg—New York, Springer-Verlag, 1972.

Das Buch gibt eine kurze, moderne und präzise Einleitung in die Theorie der Wahrscheinlichkeitsrechnung, und bietet eine sichere Basis für weitergehende Studien. Da fast alle Hilfsmittel vorbereitet sind, kann man den Text ohne höhere mathematische Vorkenntnisse studieren. Nach den einzelnen Punkten gibt es einige Aufgaben und Ergänzungen, in denen sich historische Anmerkungen und Hinweise auf weitere Literatur und auf verschiedene Anwendungsmöglichkeiten befinden. Bei der sehr präzisen und lückenlosen Betrachtungsweise sorgt Verf. darauf, daß die Schwierigkeiten der mathematischen Modellbildung gut erleutert seien. In Kapitel I werden die diskreten Wahrscheinlichkeitsräume betrachtet. (Auch die Grundformeln der Kombinatorik werden bewiesen.) Kapitel II bietet die wichtigsten Hilfsmittel der abstrakten Maß- und Integrationstheorie. Kapitel III beschäftigt sich mit den allgemeinen Wahrscheinlichkeitsräumen. Gewisse einfache stochastische Prozeße (Markoffsche Ketten, Poisson-Prozeße), weiterhin die einfachsten Gesetze der großen Zahlen, und Verteilungssätze werden auch betrachtet.

Das Buch benützt die moderne Terminologie. Die Betrachtungsweise ist gedrängt, so ist das Buch nur für solche Leser empfehlenswert, die schon eine gewiße Lesefertigkeit von mathematischen Texten besitzen.

K. Tandori (Szeged)

F. John, Partial Differential Equations (Applied Mathematical Sciences, 1) VIII+221 pages, New York—Heidelberg—Berlin, Springer Verlag, 1971.

Perhaps the following few words, taken from the preface, are most characteristic of the book which grew out of a course held by the author in 1952—53: „Though the field of Partial Differential Equations has changed considerably since those days, particularly under the impact of methods taken from Functional Analysis, the author feels that the introductory material offered here still is basic for an understanding of the subject. It supplies the necessary intuitive foundation which motivates and anticipates abstract formulations of the questions and relates them to the description of natural phenomena."

The chapter headings are: I. The single first order equation, II. The Cauchy problem for higher order equations, III. Second order equations with constant coefficients, IV. The Cauchy problem for linear hyperbolic equations in general. — There is also a list of books for further study.

The book is an excellent introduction to the classical theory of partial differential theory.

L. Pintér (Szeged)

E. A. Maxwell, Fallacies in Mathematics, 95 pages, Cambridge, University Press, 1963.

The book of Dr. Maxwell is more than an entertaining piece of reading. In the first chapter the author distinguishes among three kinds of mathematical errors: the simple mistake, causing only a momentary aberration, the howler, which leads innocently to a correct result, and the fallacy, leading by some guile to a wrong but plausible conclusion. The author investigates different types of fallacies, in particular fallacies in geometry, algebra, trigonometry, differentiation and integration. Especially interesting is the "Isosceles Triangle Fallacy" that plays an important role in subsequent chapters, one of them containing a digression on elementary geometry and the analysis of fallacies. The last three chapters are on: "Fallacy be the Circular Points at Infinity", "Some Limit Fallacies" and "Some Miscellaneous Howlers". The finding of suitable tricks together with the author's comments are very interesting and amusing reading for students of any age. Teachers of mathematics in high schools, colleges and universities will also find some useful and intriguing problems to be used in their work.

I. Szalay (Szeged)

J. L. Lions, Optimal Control of Systems Governed by Partial Differential Equations (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, 170), XI+396 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1971.

The content of the book developed from a course given at the Faculty of Sciences, University of Paris.

Practical necessities as well as investigations of optimal control theory in infinite dimensional spaces suggest in a natural way to examine models described by a family of partial differential operators. So an entirely new type of controls appear, the control by boundary values. This new character of controls and the complexity of partial differential equations in comparison with ordinary differential equations give the high difficulties of the problem.

Relying on his former book on partial differential equations the author uses a quite general technique and treats the material in a clear formulation, grouping it into five chapters according to five different areas. First, minimization of functions in a Hilbert space is presented as an introduction to infinite dimensional extremum problems. Then controls of systems governed by elliptic, parabolic and hyperbolic partial equations follow. Each of these chapters deals, besides the general problems of observability, controllability and existence of optimal controls, also with some special problems according to the character of the control system. Of course the results concerning the various types of elliptic systems are used in the sequel in the mixed problems with control on the boundary. Parabolic systems have the most interesting and ramifying relations, for instance the asymptotic behaviour of control and the connections with the Hamilton—Jacobi theory of variations. The book ends with regularization, approximation and penalization problems.

Each chapter begins with a detailed plan sketching out the scope of the chapter and closes with bibliographic notes and indications on unsolved problems.

A considerable number of comparatively simple examples complete the book and make it useful for those interested in this developing branch of mathematics and in applications of the modern theory of partial differential equations.

T. Matolcsi (Szeged)

H. H. Schaefer, Topological Vector Spaces (Graduate Texts in Mathematics), Third printing corrected, XI+294 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1972.

The book is intended to be a systematic text on topological vector spaces. At its first appearance it had a pioneering character, and it still preserved its popularity and usefulness, which justified the new edition.

After shortly recalling some prerequisits on general topology and linear algebra, topological vector spaces are treated in general. The next chapter deals with locally convex spaces including important special types of such spaces. Linear and bilinear mappings form the subject of another chapter, which has most important sections on special problems such as topological tensor products, nuclear mappings and spaces. These subjects are indispensable tools for modern functional analysis and its applications. The chapter on duality discusses $\mathfrak{S}$-topologies and reflexivity, and some other special parts of the theory, again with great emphasis on tensor products and nuclear spaces. The content of the last chapter is usually not involved in the investigations concerning topological vector spaces: this chapter has the order structures as its subject.

At the end of each chapter there is a vast amount of exercises devoted to further supplements, in particular to examples and counter examples.

The book is well organized, the presentation of material is concise but completely understandable. It may be very useful both for introduction and for reference.

T. Matolcsi (Szeged)

H. Schubert, Categories, XI+385 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1972. — DM 93,—

Categorial methods of speaking and thinking are becoming more and more widespread in mathematics. In this textbook, which only assumes that the reader has elementary prior knowledge of set theory and algebra, the central ideas of category theory are developed. The presentation of material is very concise, however definitions and notions are illuminated by a vast amount of examples and occasionally even special cases are treated to give a base for later abstractions.

After the introductory chapters including also the treatment of additive categories and categories of functors, attention is concentrated upon the concept of a representable functor and its variations: limits, colimits and adjoint functor pairs. Most of the chapters are devoted to or connected with this area of the theory. In the sequel, an elementary and then a functional exposition of algebraic structures and of Abelian and Grothendieck categories are the main subjects, followed by such other topics as fractional calculus, Grothendieck topologies and triples.

The clearly written book can serve as a good guide to those interested in this novel branch of mathematics.

T. Matolcsi (Szeged)

W. Freiberger—U. Grenander, *A Short Course in Computational Probability and Statistics* (Applied Mathematical Sciences 6), XII+155 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1971.

One of the most critical points in any presentation of applied mathematics is to preserve the unity of theory and applications. The solutions of textbook problems are often easy consequences of the theorems presented so the reader could have the impression that every problem arising in practice can be solved in five minutes, provided that one knows the adequate theorems. However, the solution of a real problem is much more difficult. From model building until numerical computation each step of the approach must be chosen carefully, otherwise the success is threatened.

The purpose of the present book is to help the reader with his work in real problems and as the authors state, the book "has been designed with the aim of making students and perhaps also faculty aware of some of consequences of modern computer technology for probability theory and mathematical statistics". It is pointed out that computational probability does not equal writing of statistical programs, and what is analytically possible need not be computationally feasible. Special attention is paid to model building and to the most effective combination of analytic and computational methods.

The material is a discussion of about 20 problem groups, the most interesting ones are as follows: random number generation, different Monte Carlo methods,i nsurance problems, growth and renewal models, Bayesian decision problems, stochastic approximation, design of experiments, analysis of variance, time series analysis, signal detection, etc. Different approaches to the problems are compared, and to each one complete APL programs are given. (In the reviewer's opinion the use of a more wide-spread programming language would have made the understanding of these programs easier.) The material presented in the book does not cover the whole area of applied probability and statistics, but the understanding of the given methods enables the reader to solve problems from other spheres of applied mathematics, too.

By its nature the book is perhaps better suited for a subject of discussions in a seminar than for individual study. To use all of its advantages, the reader must have solid knowledge in calculus, linear algebra, probability theory and statistics as well as some experience with computers.

Summing up, the present book is an excellent introduction to the ambitious applications of probability theory and statistics, and presents a new approach in the education of applied mathematics.

D. Vermes (Szeged)

H. Bühlmann, Mathematical Methods in Risk Theory (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 172), XII+210 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1970.

Insurance mathematics is one of the oldest applied mathematical disciplines, its origins go back into the 17th century. Resulting from its vigorous development in the last decades the present actuarial mathematics "undertakes to solve the technical problems of all branches of insurance, and it concerns itself particularly with the operational problems of the insurance enterprise". The mathematical basis is probability theory, but as a result of its long history, insurance mathematics has its own language, which appears somewhat strange for the non-specialist.

The present book "attempts to create a synthesis out of a selection made by the author of modern scientific publications in the field of actuarial mathematics, with the goal of presenting a unified system of thought".

The reader, who is only supposed to be familiar with the elements of probability theory, will find this book a useful introduction into the modern spheres of insurance mathematics. The chapter headings are: 1. Probability aspects of risk, 2. Risk processes, 3. The risk in the collective, 4. Premium calculation, 5. Retentions and reserves, 6. The insurance carrier's stability criteria.

D. Vermes (Szeged)

Martin Gardner's Sixth Book of Mathematical Games from Scientific American, 262 pages, San Francisco, W. H. Freeman and Co., 1972.

Martin Gardner's book is a brilliant set of mathematical puzzles, games and plays completed by historical curiosities concerning mathematics. Most of the themes are novel and quite intriguing. The book will give great delight to a wide circle of readers, in particular to students in high schools and universities, but it is a useful help for the teachers of mathematics at all levels as well. The text makes for an easy reading, the figures are clear and well arranged, and there are a few but well chosen references at the end of every section. From the topics: topology, combinatorial theory, board games, three-dimensional maze, prime numbers, graph theory, ternary system, cycloid, mathematical magic tricks, Pythagorean theorem, infinite series, lattice of integers, op art.

J. Major (Budapest)

K. Jordan, Chapters on the classical calculus of probability (Disquisitiones Mathematicae Hungaricae 4) XXV+619 pages, Budapest, Akadémiai Kiadó, 1972.

This is a translation of the original Hungarian edition that appeared in 1956. It contains a preface written by Béla Gyires and a list of Károly Jordan's works.

Károly Jordan is a classical figure in Hungarian mathematics. His results in probability theory and in the theory of finite differences are well-known. It was peculiar to his work that he was an eminent expert of applications. Although the major part of his research falls in the time before Kolmogoroff's foundation of probability theory became widely known and accepted, the book is a highly valuable and useful contribution to the literature and very enjoyable reading. The reader will find many interesting facts about the development of probability theory and philosophical disputes on the notion of probability. The author's mastery of the combinatorial methods and the presentation of various applications also increase the book's value.

Károly Tandori (Szeged)

R. Nevanlinna, Analytic Functions, 373 pages, Springer-Verlag, Berlin—Heidelberg—New York 1970.

This monograph on analytic functions is the revised translation of "Eindeutige analytische Funktionen", 2nd edition 1953 (Grundlehren der mathematischen Wissenschaften, Vol. 46).

This work coincides to a large extent with the presentation of the modern theory of single-valued analytic functions given in the author's earlier works "Le théorème de Picard—Borel et la théorie des fonctions méromorphes", Gauthier—Villars (Paris, 1929), and "Eindeutige analytische Funktionen" (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Vol. 46), Springer-Verlag (1st ed. Berlin, 1936, 2nd ed. Berlin—Göttingen—Heidelberg, 1953).

As the author says in the preface, this new edition contains some changes and additions, particularly concerning the Second Main Theorem in the Theory of Meromorphic Functions (IX. Chapter). In the derivation of this theorem the author uses a version of F. Nevanlinna's differential geometrical method, which makes the main theorem easier for access.

The chapter headings are: I. Conformal Mapping of Simply and Multiply Connected Regions, II. Solution of the Dirichlet Problem for a Schlicht Region, III. Function Theoretic Majorant Principles, IV. Relations Between Noneuclidean and Euclidean Metrics, V. Point Sets of Harmonic Measure Zero, VI. The First Main Theorem in the Theory of Meromorphic Functions, VII. Functions of Bounded Type, VIII. Meromorphic Functions of Finite Order, IX. The Second Main Theorem in the Theory of Meromorphic Functions, X. Application of the Second Main Theorem, XI. The Riemann Surface of a Univalent Function, XII. The Type of a Riemann Surface, XIII. The Ahlfors Theory of Covering Surfaces.

J. Németh (Szeged)

F. W. Stevenson, Projective planes, X+416 pages, San Francisco, W. H. Freeman and Company, 1972.

The greatest difficulty for the beginner in the study of axiomatic geometry is presented by the large number and the many kinds of axioms involved. There are essentially two kinds of axioms: some of them have an algebraic character, the others are metrical or topological. The same holds for the axiomatic theory of projective planes, cf. the standard book on the subject: G. Pickert, *Projektive Ebene* (Berlin, 1955).

The present book, as claimed in the preface, is intended to serve as a textbook for the theory of projective planes from the point of view of ring theory.

It is divided into three parts. Part 1 introduces the reader to the basic concepts and methods necessary for Parts 2 and 3. Part 2 is devoted to the classical theorems of Desargues and Pappus. (Chapters: Desarguesian planes, Pappian planes, Planes over division rings and fields, Coordinatizing planes.) In Part 3 it follows the study of non-Desarguesian planes, coordinatized by various generalized rings: planar rings with associativity, quasifields, planar nearfields, semifields, and alternative rings.

The topological aspects of the theory of projective planes are not studied.

There are many interesting examples and exercises.

P. T. Nagy (Szeged)

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